MODERN ANALYSIS OF ALTERNATING CURRENT NETWORKS

Dedicated to the memory of my
professors and friends
Hermann Weyl
and
John von Neumann

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VOLUME I

EDITORIAL LIMUSA WILEY, S. A.
MEXICO 1964

ARCOS DE BELEM 75, México, D. F.
Primera edición: 1964

Impreso en México
Talleres "Offset Técnicos Asociados, S. A."
Latinos 36, México, D. F.

Esta edición consta de 1000 ejemplares

PREFACE

This book is a reproduction in somewhat more detail of a course on Alternating Current Networks which I have developed and taught almost yearly since 1944 to sophomores and juniors in the fields of electronics and electrical engineering, together with several chapters (XIII, XVI, and part of XVIII) which correspond to lectures given now and then to graduate students, and also some extra details which are included in order that the book may later serve the student as a reference source, and for the sake of completeness.

The book starts almost from scratch and presupposes nothing beyond a working knowledge of differential & integral calculus roughly the rule to obtain the derivative of a composite function and the notion of a definite integral. The rest of the mathematics needed in the text are supplied almost completely and (except for the last part in Ch. XVIII) are quite elementary; however, they are extracted from various fields not used much ordinarily in existing treatments and so may seem a little strange at first.

This book differs from all other books on the subject of Circuit Theory mainly in that it is simultaneously elementary (except for the last part of Ch. XVIII) and completely general, this being its chief motivation. It is the result of turning back to the original work of **G. Kirchhoff** on the flow of electricity through a network (Poggendorf's Annalen der Physik, Vol. 72 [1847], pp. 497-508) and **H. Poincaré** (Analysis Situs, Journal de l'École Polytechnique, 1895; Rendiconti, Palermo, 1899 & 1904; Proceedings, London, 1900), as was done by H. Weyl (who kindly supplied me with a reprint of an article of his in the Revista Matemática Hispano-Americana, 1923), and making a critical revision of the methods employed by subsequent writers, who seem to have disregarded such work to a great extent. With the modified methods, complete proofs have been obtained for the generally accepted results which were only proved in very simple cases or not at all.

The contents of the book can best be appreciated by reading the table of contents. Volume I covers the basic parts of the subject and volume II extends the coverage to a point from which it is easy to pass into the more specialized or advanced literature. One feature of the book is that it provides easily understood clearcut rules to obtain the equations of any network whatsoever and their use. Another feature is that it shows how collections of basic elements of ever increasing complexity may be taken as the units of the network with a corresponding simplification in its combinatorial structure. The chapter XVII on three-phase networks is perhaps the only difficult one from a technical point of view; but anything

PREFACE (continued)

much less would have been too simple as to be of much use in practice. The last part of chapter XVIII on multifrequency networks is the only difficult part from a mathematical point of view, and usually I only mention the results and show their use in the undergraduate course; however, complete details and proofs are given in order that the reader need not plunge into an ocean of mathematics to obtain them.

In concluding, I wish to express my appreciation to all the Professors I had at Princeton, without whose teachings I probably could not have written this book, and to Ing. Juan Manuel Ramírez C. for his encouragement. I also wish to express my gratitude to Ing. Jesús Castillo C., who let me have his notes of my 1945 lectures, and to Ing. Wilebaldo Lara C., for his notes of my 1947 lectures.

Enrique Bustamante Ll.

Mexico City,

September, 1961.

Volume I: Essential Topics.

MODERN ANALYSIS OF ALTERNATING CURRENT NETWORKS

CHAPTER I: FUNDAMENTAL CONCEPTS

We shall assume that the reader has an elementary idea of what is meant by an electric charge, an electric current (essentially a flow of electric charge), an electric field and its intensity, a magnetic field and its intensity, and the fluxes of these quantities through any area of a given surface. These concepts are usually obtained in general courses (like physics) given before a special course on alternating currents is offered.

We recall that a voltage drop between two points, along a certain path in a given sense, is the work done by the electric field in taking a unit charge between the two points along the path in the given sense (for fixed time). A voltage rise is similarly defined as work done against the electric field, per unit charge or, in other words, as the work done by the negative of the electric field intensity. Thus a voltage rise is a voltage drop with a change of sign or, in other works, a voltage drop in the opposite sense (and viceversa).

The absolute value of a voltage rise or drop will be called simply the voltage; sometimes, however, the word „voltage" will also be used as a generic term for a voltage rise or drop.

The basic constituents of an electric circuit or network are certain things with two ends or terminals, called (two-terminal) elements. We will always assume that for any such element a sense (or reference direction) is defined given (arbitrarily) along a path through the element between the two ends, usually indicated by an attached arrow. This reference direction between the two terminals, a and b (say), will serve for both the current and the voltages.

If the attached arrow is directed (or oriented) from terminal a to terminal b then a positive value of the current through the element (at a certain instant) will mean a positive rate of flow of positive electric charge into terminal a and (since we will always assume the elements to be „lumped") an equal rate of flow out of terminal b; and a negative value of the current will mean a positive rate of flow into b and out of a. Similarly, a positive value of the voltage drop (and a negative value of the voltage rise, at a certain instant) will mean an actual voltage drop from terminal a to terminal b (i.e. positive work done by the electric field intensity), and a negative value of the voltage drop (and a positive voltage rise) in the element will mean an actual voltage rise from terminal a to terminal b (i.e. negative work done by the electric field intensity) or, in other words, an actual voltage drop from terminal b to terminal a.

We shall now proceed to review the fundamental characteristics of the basic elements, namely, resistors, condensers, coils, and the so-called voltage (or emf), and current, sources.

§1. THE RESISTOR OR RESISTANCE ELEMENT.

The essential thing about a resistor is that it is a (two-terminal) element, associated with which there is a quantity R (called its resistance), such that the voltage drop V and the current I through it (both referred to the same arbitrarily given reference direction) are related by the following equation:

$$V = RI. \tag{1}$$

If the reference direction is changed, both V and I change sign only, so that Eq. (1) remains the same. This means that this equation, which characterizes a resistor, does not depend on the assigned reference direction.

If $R \neq 0$, we can define the <u>conductance</u> G of the resistor as the reciprocal of its resistance R, and Eq. (1) can then be written:

$$I = GV. \tag{2}$$

The symbol used to represent a resistor is shown in Fig. 1(a), with an attached arrow to indicate that the reference direction for the current and voltages is to be taken from the terminal marked a to the terminal marked b.

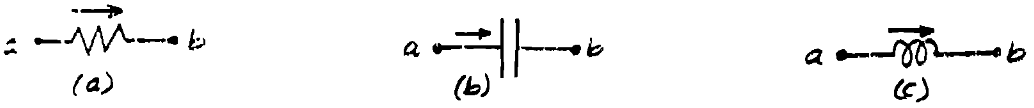


Fig. 1. Symbols used to represent resistors (a), condensers (b), and coils (c).

§2. THE CONDENSER OR CAPACITANCE ELEMENT.

The essential thing about a <u>condenser</u> (also called a <u>capacitor</u>) is that it is a (two-terminal) element, associated with which there is a quantity C (called its <u>capacitance</u>), such that the net electric charge Q having past in through one end a and out the other b up to a certain instant is related with the voltage drop V from a to b at that instant by the following equation:

$$Q = CV. \tag{1}$$

Since current is the rate at which charge passes, we have for the current going in through terminal a, and going out through terminal b, the following equation:

$$I = \frac{dQ}{dt}, \tag{2}$$

where t stands for time (measured from an arbitrary instant to be taken as the origin $t=0$). Therefore we also have:

$$Q = \int I dt = \int_0^t I dt + Q_0 \tag{3}$$

where Q_0 is the net charge having past into the element through terminal a up to $t=0$, and is called the <u>initial charge</u>. $\int_0^t I dt$ is the net charge that has past afterwards.

If $C \neq 0$, we can define a quantity S called the <u>elastance</u> of the condenser as the reciprocal of C, and then Eq. (1) can be written thus:

$$V = SQ = S\int I\,dt. \tag{4}$$

This is the equation that must be satisfied by the voltage drop and the current (referred to the same direction) between the two terminals of something in order for it to be a condenser. If the reference direction is changed, clearly both V and I only change sign and hence the Eq. (4), which characterizes a condenser, remains the same.

From Eqs. (1) and (2) we have:

$$I = \frac{d}{dt}(CV), \qquad \text{or} \qquad I = C\frac{dV}{dt} \tag{5}$$

if the capacitance is constant.

In general a condenser will have two reservoirs of electric charge (one connected to each terminal of the condenser) arranged in such a way that a charge Q in one of the reservoirs induces a charge $-Q$ in the other (e.g. two conductors separated by a dielectric). These reservoirs are called the plates of the condenser, and the symbol used to represent a condenser is shown above in Fig. 1(b) §1. The use of the word "*plates*" for the reservoirs is purely conventional and it is not intended to mean that they must be plane, even if they are so drawn in the figure.

If the reference direction is taken from the terminal a to the terminal b (as shown by the arrow in Fig. 1(b) §1) then Q must denote the charge on the plate connected to the terminal a. The charge on the other plate, connected to terminal b, shall then be $-Q$.

The mechanism by which current goes through a condenser can be explained as follows. If an electric charge Q goes in to the plate connected to the end a, this charge induces *at the same time* a charge $-Q$ on the plate connected to the end b. This means that at the same time a charge $-Q$ has gone into the element through the end b, which is equivalent to say that a charge $+Q$ has gone out through the end b. Therefore a current entering the element through one end will make an equal current go out through the other end.

§3. THE COIL OR INDUCTANCE ELEMENT.

The essential thing about a coil (also called an inductor) is that it is a (two-terminal) element associated with which there is a function Ψ, called its flux linkage, such that the voltage drop between its terminals (in any one of the two possible reference directions) is given by the time derivative of Ψ, that is,

$$V = \frac{d\Psi}{dt}, \qquad (1)$$

where t denotes the time.

Usually an idealized coil is conceived as several turns of a single isolated infinitely conducting path with two ends wound along the boundary of a two-sided simply connected surface. An arbitrarily assigned reference direction from one end to the other along the path will define (in the usual way) an associated positive direction through the two-sided surface, as shown in Fig. 1. Suppose a magnetic flux ϕ crosses the surface in the associated positive direction. Completing a closed path with a line (shown dotted) touching each consecutive turn in reverse order and applying Faraday-Neumann's law on electromagnetic induction to each turn closed with the corresponding part of the dotted line we infer that the work done in the assigned reference direction (for fixed time) by the electric field on a unit charge taken around such paths will all be equal (except for a constant factor taken =1) to the negative time derivative of ϕ; hence, if N is the total number of turns in the complete path, we obtain by addition that the total work done along the complete path in the assigned reference direction will be $-Nd\phi/dt$. But the electric field vanishes within the infinitely conducting path; hence the total work will be done entirely along the dotted line from b to a. Changing signs we will then have for the voltage drop from a to b:

$$V = \frac{dN\phi}{dt}; \qquad (2)$$

thus such a thing will be a coil with the flux linkage $\Psi = N\phi$.

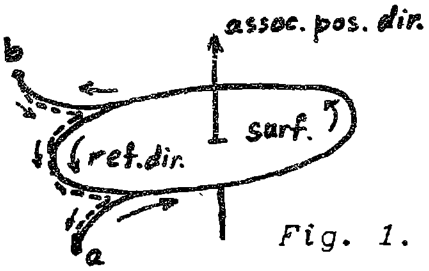


Fig. 1.

The symbol used to represent a coil is shown in Fig. 1 (c) of §1, with an attached arrow to indicate that the reference direction for the currents and voltages is arbitrarily chosen from end a to end b. If the reference direction is changed then the sign of V is changed, but since the associated positive direction is also changed, the sign of ϕ is also changed; thus eq. (2) remains the same, independent of the assigned reference direction.

In case the coil is magnetically isolated, the magnetic flux ϕ and the flux linkage Ψ depend only on the current I through it (ignoring residual effects) and so, by the rule of differentiating a composite function, we will then have in such cases that

$$\frac{d\Psi}{dt}=\frac{d\Psi}{dI}\frac{dI}{dt} \quad \text{and} \quad \frac{d\phi}{dt}=\frac{d\phi}{dI}\frac{dI}{dt}.$$

Therefore Eqs. (1) and (2) become:

$$V=\frac{d\Psi}{dI}\frac{dI}{dt}=N\frac{d\phi}{dI}\frac{dI}{dt};$$

so that, if we put:

$$L=\frac{d\Psi}{dI}=N\frac{d\phi}{dI}, \tag{3}$$

we will have:

$$V=L\frac{dI}{dt}. \tag{4}$$

The quantity L defined by Eq. (3) is known as the (coefficient of) inductance of the coil, and since the magnetic flux in the associated positive direction increases with the current in the reference direction we shall always have L ≧ 0, by Eq. (3).

When L ≠ 0, the invertance (or inverse inductance) Γ of the coil is defined as the reciprocal of L, and we also have $\Gamma \geqq 0$. In terms of Γ, Eq. (4) can be put into the following form:

$$I=\int \Gamma V dt = \Gamma \int V dt = \Gamma \int_0^t V dt + I_0, \qquad (\text{if } \Gamma \text{ is constant}) \tag{5}$$

where I_0 is the current in the coil when t = 0 and is called the initial current.

Problem 1. Show that in a magnetically isolated coil of constant inductance and with no magnetic flux through it when there is no current we have $\phi=\frac{L}{N}I$ and $\Psi=LI$, and $I=\Gamma\Psi=\Gamma N\phi$.

Problem 2. If a single conducting path (like a fine insulated wire) is wound with N_1 turns around the boundary of a surface S_1 and with N_2 turns around the boundary of a surface S_2 and so on till finally it is wound with N_n turns around the boundary of a surface S_n, and if ϕ_k (k=1,2,...,n) denotes the magnetic flux through S_k in the associated positive direction through it, then the flux links of the wire with ϕ_k are $N_k\phi_k$ and the voltage drop along the part of the wire around the boundary of S_k is $\frac{dN_k\phi_k}{dt}$. Hence show that the total voltage drop along the wire is: $V=\sum_{k=1}^{n}\frac{dN_k\phi_k}{dt}=\frac{d}{dt}\sum_{k=1}^{n}N_k\phi_k$. Thus such a winding will be a coil with the flux linkage: $\Psi=N_1\phi_1+N_2\phi_2+\cdots+N_n\phi_n$.

Let us now consider a system of n coils, numbered 1,2,...,n, to each of which a reference direction has been assigned arbitrarily. Let N_k be the number of turns in coil $k (=1,2,\dots,n)$, ϕ_k the magnetic flux through it in the associated positive direction, Ψ_k its flux linkage, and V_k the voltage drop between its terminals in the assigned reference direction. Then by Eqs. (1) and (2) we have:

$$V_k = \frac{d\Psi_k}{dt} = N_k \frac{d\phi_k}{dt}. \qquad (k=1,2,\cdots,n) \qquad (6)$$

In this book we shall only consider the case of stationary coils, in which the magnetic fluxes ϕ_k are produced alone by the currents in the coils. We shall then have (by the rule of differentiating a function of various functions):

$$\frac{d\Psi_k}{dt} = \sum_{l=1}^{n} \frac{\partial \Psi_k}{\partial I_l}\frac{dI_l}{dt} \quad \text{and} \quad \frac{d\phi_k}{dt} = \sum_{l=1}^{n} \frac{\partial \phi_k}{\partial I_l}\frac{dI_l}{dt}.$$

Therefore Eq. (6) becomes:

$$V_k = \sum_{l=1}^{n} \frac{\partial \Psi_k}{\partial I_l}\frac{dI_l}{dt} = N_k \sum_{l=1}^{n} \frac{\partial \phi_k}{\partial I_l}\frac{dI_l}{dt};$$

so that if we put:

$$L_{kl} = \frac{\partial \Psi_k}{\partial I_l} = N_k \frac{\partial \phi_k}{\partial I_l}, \qquad (k,l=1,2,\cdots,n) \qquad (7)$$

we get:

$$V_k = \sum_{l=1}^{n} L_{kl}\frac{dI_l}{dt}. \qquad (k=1,2,\cdots,n) \qquad (8)$$

The n^2 quantities L_{kl} $(k,l=1,2,\cdots,n)$ defined by Eq. (7) are called the (coefficients of) mutual inductances of (or between) the coils of the system. L_{kl} is called the mutual inductance of coil k with coil l; but when $k=l$ one speaks more frequently of the (coefficient of) self inductance (or simply inductance) of the coil k (instead of the mutual inductance of coil k with itself). In practice L_{kk} is usually denoted simply by L_k and, as in the case of a single coil, we always have $L_k \geq 0$, for all k. When $L_{kl} = 0$ for all $l \neq k$, coil k shall be magnetically isolated from the other coils and its self inductance will obviously be the same as its inductance as defined before for an isolated coil.

Since the flux linkages Ψ_k depend on the relative positions of the coils and on the permeability of the intervening medium, so shall the (self and mutual) inductances L_{kl}.

Problem 3. For a system of coils, some of which are moving as a rigid body around an axis, fixed with respect to the others, at an angular velocity $\omega = \frac{d\vartheta}{dt}$, the fluxes ϕ_k and the flux linkages Ψ_k (k=1,2,...,n) are functions of the currents I_k (k=1,2,...,n), of the angle ϑ, and perhaps of the time t (explicitly). Show that the voltage drops in the coils are given by the following equations:

$$V_k = \sum_{l=1}^{n} L_{kl}\frac{dI_l}{dt} + \omega\frac{\partial \Psi_k}{\partial \vartheta} + \frac{\partial \Psi_k}{\partial t}. \qquad (k=1,2,\cdots,n)$$

(When the fluxes are not explicit functions of the time the last term may, of course, be omitted.) These equations find application in electrical machinery.

Eqs. (8) (for k=1,2,...,n) can be used to give an abstract formulation of the idea of a system of n stationary coils in which the flux linkages are produced alone by the currents in them. We say that n things, each with two terminals, associated with which there are n^2 quantities L_{kl} $(k,l=1,2,\cdots,n)$ (to be called their self and mutual inductances), constitute a system of n stationary coils, when the currents I_k and the voltage drops V_k (k=1,2,...,n) in arbitrarily assigned reference directions between their terminals are related by the system of equations (8).

Let us now suppose that the determinant, $\det(L_{kl})$, of the (self and mutual) inductances is not zero (see note 2, below). Then Eqs. (8) can be solved for the derivatives of the currents, in terms of the voltage drops, as follows:

$$\frac{dI_k}{dt} = \sum_{l=1}^{n} \Gamma_{kl} V_l, \qquad (k=1,2,\cdots,n) \qquad (9)$$

where Γ_{kl} is the cofactor of L_{lk} (denoted cof L_{lk}) in the determinant of the (self and mutual) inductances, divided by the determinant. (In the older literature, Γ_{kl} is sometimes defined as the minor of L_{lk}, divided by the determinant; had it been so defined here, the factor $(-1)^{k+l}$ would have had been introduced in the summation, which would be rather clumsy.)

Eqs. (9) may at once be written as follows:

$$I_k = \sum_{\ell=1}^{n} \int \Gamma_{k\ell} V_\ell \, dt = \sum_{\ell=1}^{n} \int_0^{t} \Gamma_{k\ell} V_\ell \, dt + I_{0k}, \qquad (k=1,2,\dots,n) \tag{10}$$

where I_{0k} is the initial current in the coil k (i.e. the current when t=0). If all the (self and mutual) inductances are constant, these equations may be put into the following familiar form:

$$I_k = \sum_{\ell=1}^{n} \Gamma_{k\ell} \int V_\ell \, dt. \qquad (k=1,2,\dots,n) \tag{11}$$

The quantities:

$$\Gamma_{k\ell} = \frac{\text{cof } L_{\ell k}}{\det(L_{k\ell})} \qquad (k,\ell=1,2,\dots,n) \tag{12}$$

will be called the mutual invertances or (inverse mutual inductances) of the system of coils. $\Gamma_{k\ell}$ is called the mutual invertance of coil k with coil ℓ, in the presence of the other coils of the system; but if $k=\ell$, it will be called the self invertance (or inverse (self) inductance) of coil k, in the presence of the other coils. In practice Γ_{kk} will be denoted simply by Γ_k, and as a direct consequence of Lenz's law we have $\Gamma_k \geq 0$ for all k; because if all the coils are short-circuited except the generic coil k then the current in this coil must increase in the direction of the voltage drop in order to oppose the changing flux.

It should be noted that, in spite of the name, $\Gamma_{k\ell}$ is not in general the reciprocal of $L_{k\ell}$. We may even have, for example, $L_{k\ell}=0$ and $\Gamma_{k\ell} \neq 0$ for some k and ℓ (and viceversa). In general, two coils k and ℓ with $L_{k\ell}=0$ may have $\Gamma_{k\ell} \neq 0$ if there is a chain of magnetically coupled coils linking them.

In a system of coils where only a few mutual inductances are not zero, it is customary to indicate the non-zero ones by drawing arcs between the corresponding coils (see Fig. 2, p. 14). The same is done to indicate the non-zero invertances when Γ's are used to characterize the system of coils. But naturally the set of arcs when Γ's are used is not in general the same as the set of arcs when L's are used, even for the same system of coils. Therefore it shall be important in practice to mention explicitly which system of arcs is being used, especially if changes from L's to Γ's are to be made (or viceversa).

Note 1. If the coils are windings immersed in a medium of constant permeability throughout all space, it can be shown that the L_{kl} are all constant and that $L_{lk} = L_{kl}$ (for all k and l). In fact, these equations can be shown to hold if we only assume that all the (self and mutual) inductances are constant. (See L.Page & N.Adams: Principles of Electricity, van Nostrand, 1931, p.367 & p.378.)

Note 2. In practice, the determinant, $\det(L_{kl})$, for a system of coils with constant (self and mutual) inductances, is never zero. Because in this case it can be shown that the magnetic energy of the system (which is never negative, and is zero only if the magnetic field vanishes everywhere, since it is, essentially, the integral of the square of the magnetic field intensity taken throughout all space) is given by the following quadratic form (=homogeneous polynomial of the second degree):

$$\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} L_{kl} I_k I_l .$$

This means that this quadratic form is positive definite, and this implies that the determinant of the inductances is never zero. For if $\det(L_{kl})=0$ then I_l exist, not all 0, such that $\sum_l L_{kl} I_l = 0$ for all k, and so $\sum_k \sum_l L_{kl} I_k I_l = 0$.

Note 3. The coupling coefficient, K_{kl}, between any two coils k and l of a system of coils, is defined as the positive square root of the quantity $|L_{kl}L_{lk}/L_k L_l|$. (It can be shown (cf. Prob.3 §9) that L_{kl} and L_{lk} are always of the same sign, so that the absolute value in this definition is not really necessary.) As an immediate consequence we have $K_{kl} = K_{lk}$. Substituting the inductances according to Eqs.(7), we get for the coupling coefficient:

$$K_{kl}^2 = \frac{|L_{kl}L_{lk}|}{L_k L_l} = \left| N_k \frac{\partial\phi_k}{\partial I_l} N_l \frac{\partial\phi_l}{\partial I_k} \Big/ N_k \frac{\partial\phi_k}{\partial I_k} N_l \frac{\partial\phi_l}{\partial I_l} \right| = \left| \frac{d_{I_l}\phi_k}{d_{I_l}\phi_l} \cdot \frac{d_{I_k}\phi_l}{d_{I_k}\phi_k} \right| , \qquad (13)$$

where $d_{I_l}\phi_k = \frac{\partial\phi_k}{\partial I_l} dI_l$ denotes the partial differential of ϕ_k with respect to I_l, etc. Now, any given increment dI_l of the current in coil l, alone, will produce a differential in the flux ϕ_k in coil k at most equal, in absolute value, to the differential in the flux ϕ_l in coil l itself (and in reality the equality can never be attained)

Therefore we will have $|d_{I_l}\phi_k / d_{I_l}\phi_l| \leq 1$; and similarly $|d_{I_k}\phi_l / d_{I_k}\phi_k| \leq 1$. Consequently, by Eq. (13) we have $K_{kl}^2 \leqq 1$, or in other words,

$$|L_{kl} L_{lk}| \leq L_k L_l . \tag{14}$$

When the mutual inductances are symmetric (i.e. when $L_{kl} = L_{lk}$), as when all the inductances are constant, we have

$$L_{kl}^2 \leqq L_k L_l . \tag{15}$$

Note 4. The (partial) leakage flux (which is sometimes also called the stray flux), of coil k with (respect to) coil $l \neq k$, for given values of the currents $I_1, \ldots, I_n$ in the coils of the system, is defined as that part of the flux ϕ_k through coil k which does not link with coil l. If we denote it by ϕ'_{kl}, we shall have:

$$\frac{\partial \phi'_{kl}}{\partial I_k} = \frac{\partial \phi_k}{\partial I_k} - \left|\frac{\partial \phi_l}{\partial I_k}\right|$$

Hence, by Eqs. (7), we get:

$$\frac{\partial \phi'_{kl}}{\partial I_k} = \frac{L_{kk}}{N_k} - \frac{|L_{lk}|}{N_l}$$

In analogy with the definition of an inductance (see Eqs. 7), the leakage inductance (which is sometimes called the stray inductance) of coil k with (respect to) coil l is defined to be the following quantity:

$$L'_{kl} = N_k \frac{\partial \phi'_{kl}}{\partial I_k} = L_{kk} - \frac{N_k}{N_l} |L_{lk}| . \tag{16}$$

The term $\frac{N_k}{N_l}|L_{lk}|$, which is the difference between the self inductance L_{kk} of coil k and the stray inductance of coil k with coil l, may be called the working (or useful) inductance, $L_{k(l)}$ (say), of coil k with (respect to) coil l.

Problem 4. Show that the coupling coefficient, $\frac{|L_{kl} L_{lk}|}{L_{k(l)} L_{l(k)}}$, corresponding to the working inductances of two coils (i.e. their self inductances diminished by their stray inductances with each other) is always unity.

Problem 5. If the mutual inductance $|L_{lk}|$ is a bilinear function, $(N_k N_l / \mathcal{R}_{kl})$, of the coil turns N_k and N_l, where $\mathcal{R}_{kl}$ is essentially the "reluctance" of the flux path linking coils k and l, and the self inductance $L_k = L_{kk}$ is quadratic in N_k, say $N_k^2 / \mathcal{R}_k$, where $\mathcal{R}_k$ is essentially the „reluctance" of the flux path linking coil k; show that the working inductance and the stray inductance of coil k with coil ℓ are quadratic in N_k. Also show that stray inductance is given by:

$$L'_{k\ell} = L_k\left[1-\left(\frac{\mathcal{R}_k}{\mathcal{R}_\ell}\cdot\frac{L^2_{\ell k}}{L_k L_\ell}\right)^{\frac{1}{2}}\right] = L_k\left[1-\frac{N_k}{N_\ell}\frac{|L_{\ell k}|}{L_k}\right]. \tag{17}$$

The quantity in brackets, namely the coefficient of L_k, may well be called the stray coefficient, (or the leakage coefficient) of coil k with (respect to) coil ℓ. Finally, if $L_{k\ell} = L_{\ell k}$ and $\mathcal{R}_k = \mathcal{R}_\ell$, show that:

$$L'_{k\ell} = L_k(1-K_{k\ell}), \tag{18}$$

and hence that, in such a case, the stray coefficient of coil k with coil ℓ is the complement to unity of the coupling coefficient, and that the working inductance of coil k with respect to coil ℓ is equal to $K_{k\ell}L_k$, the product of the coupling coefficient by the self inductance of coil k.

Problem 6. Show that if two coils, of self inductances L_1 and L_2, with a mutual inductance M, have a coupling coefficient $K > 0$, then two coils with self inductances KL_1 and KL_2, and the same mutual inductance M, have a coupling coefficient = 1.

Problem 7. Show that if two coils of self inductances L_1 and L_2, with a mutual inductance M, have a coupling coefficient $K > 0$, then two coils with self inductances αKL_1 and KL_2/α $(\alpha > 0)$, and the same mutual inductance M, have a coupling coefficient = 1. In particular, then, we have the pairs: K^2L_1 and L_2 (when $\alpha = K$); L_1 and K^2L_2 (when $\alpha = 1/K$); and, if $K \neq 1$, the pair: $K(1-K)L_1$ and $KL_2/(1-K)$ (when $\alpha = 1-K$).

Problem 8. If we define, for a system of coils 1,2,...,n, the (total) leakage, or stray, flux, ϕ'_k, of coil k, as that part of ϕ_k which does not link with any other coil of the system, show that:

$$\frac{\partial\phi'_k}{\partial I_k} = \frac{\partial\phi_k}{\partial I_k} - \sum_{\ell\neq k}\left|\frac{\partial\phi_\ell}{\partial I_k}\right|.$$

Hence, if we define the (total) leakage, or stray, inductance of coil k to be: $L'_k = N_k\frac{\partial\phi'_k}{\partial I_k}$, show that:

$$L'_k = L_k - \sum_{\ell\neq k}\frac{N_k}{N_\ell}|L_{\ell k}|. \tag{19}$$

(The concepts and relations established in the preceding notes and problems find wide applications in the study of transformers and machinery.)

Problem 9 Assume that all the coils of a system of coils are left open (i.e. with their terminals loose), except coil *k* which is short-circuited (i.e. its terminals are connected together), and coil l which is connected to a battery supplying a constant voltage *E*. Show that:

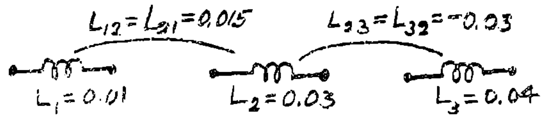
$$E = \left(L_l - \frac{L_{kl}L_{lk}}{L_k}\right)\frac{dI_l}{dt}.$$

From this equation deduce Eq. (14), by showing that on applying the voltage *E* to coil l (by closing an inserted switch, for example), the current in this coil starts increasing. The coefficient $L_l - \frac{L_{kl}L_{lk}}{L_k} = L_l(1-K_{kl}^2)$ in the above equation is known (especially in the older literature) as the *leakage inductance of the pair of coils k and l*, referred to the coil l, and should be distinguished from the leakage inductance of coil *k* with (respect to) coil l (or viceversa). On the other hand, the quantity $(1-K_{kl}^2)$ is almost always called the *leakage coefficient of the pair of coils k and l* (by definition), and should be distinguished from the leakage, or stray, coefficient of coil *k* with (respect to) coil l (or viceversa), as defined previously (in problem 5), however it could also be called the *dispersion coefficient* in an effort to avoid confusion.

Problem 10. Consider a system of *n* coils (with arbitrarily assigned reference directions). Let ϕ_{kl} denote the contribution of the flux produced by the current in coil l to the total flux through coil *k* (in the associated positive direction), and assume that this contribution depends only on the current I_l producing it (for all *k* and $l = 1, 2, \ldots, n$). Then we shall have $\phi_k = \phi_{k1}(I_1) + \cdots + \phi_{kn}(I_n)$. Show that in such a system, the inductances are given by $L_{kl} = N_k\, d\phi_{kl}/dI_l$. In particular, this will be the situation when the system of coils is immersed in a medium of constant permeability (throughout all space), and then the inductances are constant. In this case show that the flux linkages are given by:

$$\psi_k = N_k\phi_k = \sum_{l=1}^{n} L_{kl} I_l + \mathrm{Const}_k. \qquad (k = 1, 2, \ldots, n). \qquad (20)$$

Problem 11. Compute the self and mutual invertances (inverse inductances) for the system of coils shown in the figure 2. Is $\Gamma_{13} = 0$?



$(L_{13}=L_{31}=0.)$

Fig. 2.

Compute the various coupling and leakage (stray and dispersion) coefficients, as well as the leakage inductances. Also, assuming equal numbers of turns in each coil, compute the various (total and partial) stray (or leakage), and working, inductances. Express the flux linkages Ψ_k (k=1,2,3) in terms of the currents as variables, assuming that there are no fluxes when all the currents are zero.

Problem 12. Show that for a system of coils with constant (self and mutual) inductances we have: $\Gamma_{kl}=\Gamma_{lk}$, for all k and l. [Hint: $L_{kl}=L_{lk}$.]

Problem 13. By considering all coils of a system short-circuited except coil k, which is left open, and coil l, which is connected to a battery, show that $|\Gamma_{kl}\Gamma_{lk}|\leq\Gamma_k\Gamma_l$ Hence, if $\Gamma_{kl}=\Gamma_{lk}$ then $\Gamma_{kl}^2\leq\Gamma_k\Gamma_l$. (Cf. Prob. 9)

Problem 14. For a system of coils with constant (self and mutual) inductances and with no fluxes when there are no currents, we have:

$$\Psi_k=N_k\phi_k=\sum_{l=1}^{n}L_{kl}I_l. \quad (k=1,2,\cdots,n) \tag{20'}$$

Hence show that: $I_k=\sum_{l=1}^{n}\Gamma_{kl}\Psi_l=\sum_{l=1}^{n}N_l\Gamma_{kl}\phi_l.$ $(k=1,2,\cdots,n)$ (21)

Also show (using the notation of problem 10) that: $\phi_{kl}=L_{kl}I_l/N_k$,

and that for the coupling coefficient we therefore have:

$$K_{kl}^2=\left|\frac{L_{kl}\,L_{lk}}{L_{kk}\,L_{ll}}\right|=\left|\frac{\phi_{kl}\,\phi_{lk}}{\phi_{kk}\,\phi_{ll}}\right|$$

Thence, since ϕ_{kl} is only a part of ϕ_{ll} , and ϕ_{lk} is only a part of ϕ_{kk} , infer that $K_{kl}^2\leq 1$. This is the way this result (obtained in all generality in Note 3, p.10) is usually established for the case of constant (self and mutual) inductances.

Problem 15. If the det$(\Gamma_{kl})\neq 0$, show that:

$$L_{kl}=\mathrm{cof}\,\Gamma_{lk}/\det(\Gamma_{kl}).$$

Problem 16. When the coupling coefficient between two coils is unity, they are said to have perfect coupling. In such a case

[illegible]

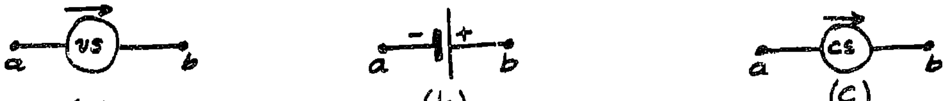
§4. THE VOLTAGE SOURCE.

The essential thing about a voltage source (abbreviated *vs*) is that it is a (two-terminal) element, associated with which, relative to an arbitrarily assigned reference direction, there is a function, E(t), of the time *t*, called its electromotive force (or value), such that this thing is compelled to have a voltage rise = E(t) between its terminals in the reference direction at the time *t*, for all *t*. Thus the voltage (rise or drop) between the terminals of a voltage source is obliged to follow some prescribed function of the time, and this must happen no matter what current passes through the source, as long as its terminals are not short-circuited (i.e. connected together). In the literature, such a contrivance is also called an „*infinite bus*."

Practical voltage sources are usually called (voltage) generators. Such are, for example, the ordinary battery (a perfect battery would be a source of constant value), and the common alternator.

In general, practical voltage sources are not true voltage sources. However they can usually be approximated very closely by an appropriate combination of basic elements. Thus a battery is well represented by a constant voltage source (i.e. a *vs* of constant value) connected in series with a resistor (its „internal resistance"). A simple alternator is well represented by a *vs* in series with a resistor and a coil (its „internal resistance and inductance"), for many cases.

The symbol used to represent a voltage source is shown in Fig.1(a). The arrow is attached to indicate the reference direction in which the value E(t) of the source is to be considered as the voltage rise (from terminal *a* to terminal *b* in the figure). When the value of a voltage source is constant, the symbol shown in Fig.1(b) is sometimes used. (This symbol is also used to represent a battery.) The + and - signs are attached to indicate the sense in which the voltage rise is positive (from end a to end b in the figure); or in other words, the sense from the terminal with the - sign to the terminal with the + sign is taken as the reference direction. At any instant at which the value, E(t), of a source is negative, there is a negative voltage rise in the reference direction which means that the voltage actually drops in that direction. If the reference direction for a voltage source is changed then the sign of its value must also be changed.



(a) (b) (c)

Fig. 1. Symbols used to represent sources.

§ 5. THE CURRENT SOURCE.

The essential thing about a current source (abbreviated cs) is that it is a (two-terminal) element, associated with which, relative to an arbitrarily assigned reference direction, there is a function, I(t), of the time t, called its current (or value), such that at the time t the current through it in the reference direction is compeled to be = I(t), for all t. Thus the current through a current source is obliged to follow some prescribed function of the time, and this must happen no matter what voltage appears between its terminals, as long as they are not left „open-circuited.”

The symbol used to represent a current source is shown in Fig. 1 (c) §4. The attached arrow is intended to indicate the sense between the terminals of the element to be taken as the reference direction (from end a to end b in the figure). A negative value of I(t) at a certain instant will mean a flow of negative charge in the reference direction or, what is equivalent, a flow of positive charge in the opposite direction, at the instant in question.

If the reference direction for a current source is changed then the sign of its value must also be changed, or, in other words, the value of a current source in an opposite reference direction is -I(t).

§6. PASSIVE AND ACTIVE ELEMENTS.

There is a distinct difference between the first three basic elements we have considered in §§1 to 3, and the last two we considered in §§4 and 5. To be more explicit, in the cases of the resistance, capacitance, and inductance, elements there are explicit relations between the voltages and currents, and these hold no matter how they are connected in a circuit. In other words, the relations between voltages and currents for these elements are invariant to changes in their connections. On the other hand, in the cases of voltage and current sources, the situation is quite different. A definite relation between the voltage and the current in the source shall only be present once the source is connected in a definite way, and this relation shall depend entirely on the way the source is connected.

The first three element, namely, resistors, condensers and coils are called passive elements, and the values associated with passive elements, namely, resistances, elastances, (self and mutual) inductances, conductances, capacitances, and (self and mutual) invertances (or inverse inductances), are called network (or circuit) parameters.

The other two basic elements, namely, voltage sources and current sources, are called active elements, and the values of active elements are called „activation, or driving, forces" (or sometimes „exciting functions").

In short, an element shall be a passive or active element according to whether or not (respectively) there is a relation between voltages and currents which does not depend on the way the element is connected, that is, which is invariant to changes in the connections.

§7. POWER AND ENERGY.

The (electric) power absorbed by an element with a current $I(t)$ through it in a certain direction and a voltage drop $V(t)$ between its terminals in the same direction can be obtained as follows. Let an electric charge ΔQ be carried through the element in an interval Δt of time. Then since $V(t)$ is the work done, for fixed time t, by the electric field in carrying a unit charge through the element, the work done on the charge ΔQ is $\bar{V}\cdot\Delta Q$, where $\bar{V}$ is the average, or mean value, of $V(t)$ in the interval in question. Consequently, the average rate at which energy is consumed by the element is $\bar{V}\cdot\Delta Q/\Delta t$. The limit of this when $\Delta t \to 0$, that is, the instantaneous rate at which energy is consumed by the element (from the circuit) is known as the absorbed power. If we denote this by $W(t)$, we will therefore have:

$$W(t) = \lim_{\Delta t\to 0} \bar{V}\cdot\frac{\Delta Q}{\Delta t} = V(t)\cdot\frac{dQ}{dt} = V(t)\cdot I(t). \qquad (1)$$

Had the voltage rise $E(t)$ in the direction of the current been used instead of the voltage drop $V(t)$, the result, namely, $E(t)\cdot I(t)$, would be the (electric) power given out by the element (to the circuit), known as the supplied power.

We need not continue to stress the distinction between absorbed and supplied power, because a negative absorbed power will mean a supplied power, and a negative supply will mean an absorption. In general, when dealing with passive elements we will consider absorbed power, and when dealing with active elements we will consider supplied power.

Once the power is known, it is easy to compute the electric energy taken in any interval (from t_1 to t_2) of time, namely: $U = \int_{t_1}^{t_2} W(t)dt$.

In a resistor the absorbed energy is transformed into heat. In a condenser it is transformed into energy of (or stored in) an electric field, and in a coil, into energy of (or stored in) a magnetic field.

Problem 1. Show that the power absorbed in a resistance is:

$$W = RI^2 = GV^2. \tag{2}$$

Problem 2. Show that for a condenser, the absorbed power is:

$$W = \frac{1}{2} S \frac{d}{dt} Q^2, \tag{3}$$

and if its capacitance is constant:

$$W = \frac{d}{dt}\left(\frac{1}{2} CV^2\right). \tag{3'}$$

Problem 3. Show that for an isolated coil, the absorbed power is:

$$W = \frac{1}{2} L \frac{d}{dt} I^2, \tag{4}$$

and if the inductance is constant:

$$W = \frac{d}{dt}\frac{(N\Phi)^2}{2L} = \frac{d}{dt}\left(\frac{1}{2}\Gamma \Psi^2\right). \tag{4'}$$

Problem 4. In a system of n coils show that the power absorbed by the coil k is:

$$W_k = V_k I_k = \sum_{l=1}^{n} L_{kl} I_k \frac{dI_l}{dt}. \tag{5}$$

Then show that the total power absorbed by the system of n coils is:

$$W = \frac{d}{dt} \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} L_{kl} I_k I_l, \tag{5'}$$

if all the inductances are constant. Would this last result be true for any system where $L_{kl} \neq L_{lk}$, for some k and l ?

Problem 5. The mean value, or average, of a periodic function f(t), of period T(≠0), is defined as the mean value of f(t) in the interval $0 \leq t \leq T$, namely: $\frac{1}{T}\int_0^T f(t)\,dt$. Show that the mean value of the power W = VI in the case that $V = A\cdot\sin(\omega t + \alpha)$ and $I = B\cdot\sin(\omega t + \beta)$, with $\omega \neq 0$, is: $\frac{1}{2}AB\cdot\cos(\beta - \alpha)$. [Take $T = \frac{2\pi}{\omega}$, although $T = \frac{\pi}{\omega}$ will do.]

Problem 6. Show that for a constant resistance in which the current is $I = A\cdot\sin(\omega t + \alpha)$, the average power is $RA^2/2$. Also show that for a resistor of constant conductance G, in which the voltage drop is $V = A\cdot\sin(\omega t + \alpha)$, the average power is $GA^2/2$. (Here, of course, we assume that $\omega \neq 0$.) Notice that the average power is independent of the angle α. Notice also that a constant current or constant voltage (as the case may be) which would yield the same result is $|A|/\sqrt{2}$.

For this reason this value is known as the *effective value* of $A\sin(\omega t+\alpha)$. It can easily be shown that this value is the positive square root of the mean value of the square of the sine function. For this reason the effective value of a sine function is also known as its *root-mean-square* (abbreviated *rms*) *value*. (See Ch. XII, §2, p. 294.)

Problem 7. Show that the absolute value of an inductance $L_{k\ell}$ can be obtained by applying to coil k a sine voltage, $A \sin(\omega t+\alpha)$, and placing a voltmeter between the terminals of coil ℓ in order to measure the effective value of the induced voltage (leaving all other coils open-circuited), and inserting an ammeter in coil k in order to measure the effective value of the current in it. When $k=\ell$, this gives the self inductance, L_k, of coil k. (Assume all currents and voltages to be sine functions of the same kind as the applied voltage.)

Problem 8. Show that the invertance (inverse inductance) $\Gamma_{k\ell}$ can be obtained, in absolute value, by applying to coil ℓ a voltage source of (known) sinusoidal value, and inserting in coil k an ammeter in order to measure the effective value of the current through it, and short-circuiting all coils except coil ℓ. (Assume all the currents to be sine functions of the same kind as the applied voltage.) When $k=\ell$, this gives the self invertance of coil k (in the presence of the other coils).

§8. THE MKSC SYSTEM OF UNITS.

So far nothing has been said about the units for the various quantities we have spoken about. It should be understood that any convenient consistent system of units may be used. However, we shall be more specific about the matter, and we shall now consider a definite system called the *meter-kilogram(mass)-second-coulomb* (abbreviated *MKSC*) *system*, one of the systems due to G. Giorgi (Unita Razionali di Elettromagnetismo, Atti dell' Assoc. Elet. Italiana, 1901). This system is now being used in many modern books on engineering (rationalized or not).

In this system, a unit of lenght (the meter, abbreviated *mt*), a unit of mass (the kilogram, abbreviated *kg* or *kgm*), a unit of time (the second, abbreviated *sec*), and a unit of electric charge (the coulomb, abbreviated *coul* or *clb*) are taken as fundamental units.

The unit of area is the *mt^2*, the unit of volume is the *mt^3*, the unit of speed and velocity is the *mt/sec*, and the unit of acceleration is the (mt/sec)/sec or *mt/sec^2*, by definition. From Newton's law, the unit of force is the unit of mass by the unit of acceleration, namely, the $\text{kg(mt/sec}^2)$ or $\text{kg}\cdot\text{mt/sec}^2$, and this unit is called the *newton*. The unit of work and energy is (by definition) the unit of force by the unit of length, that is, the *newton·meter*, which is the same thing as the *$\text{kg}\cdot\text{mt}^2/\text{sec}^2$*, and is called the *joule*. The unit of power is the *joule/sec*, known as the *watt*, since power is the rate at which work is done.

Since an electric current is measured by the rate of flow of electric charge , the unit of current is the coulomb/sec, known as the *ampère* (abbreviated *amp*). Also, by definition, a voltage, a potential difference, and an electromotive force, are work per unit charge, so that the unit of these quantities is the joule/coulomb, which is known as the *volt*.

By §1, the unit of resistance is the unit of voltage divided by the unit of current, i.e. the volt/ampère, known as the *ohm*; and the unit of conductance is the ampère/volt or *inverse ohm*, known as the *mho* (*ohm* written backwards) or siemens. By §2, the unit of capacitance is the coulomb/volt, which is known as the *farad*; and the unit of elastance is the volt/coulomb, or *inverse farad*, known as the *daraf* (*farad* written backwards). By §3, the unit of inductance is the volt/(amp/sec) = volt.sec/amp, which is known as the *henry*; and the unit of invertance (or inverse inductance) is the amp/(volt.sec), or *inverse henry*, known as the *yrneh* (*henry* written backwards).

Problem 1. Show that a volt is a $(kg.mt^2)/(coul.sec^2)$.

Problem 2. In §3 (p.5) we mentioned that the constant factor of proportionality between the electromotive force around a closed path and the rate of decrease of the magnetic (induction) flux through it (in the associated positive direction), expressing the law of Faraday-Neumann on electromagnetic induction, was to be taken = 1. In the MKSC-system this constant is also taken to be dimensionless (i.e. an abstract number). From this show that the unit of magnetic flux (in the MKSC-system) is the volt.sec = $(kg.mt^2)/(coul.sec)$, which is known as the *weber*. (Note. The unit of flux linkages is conventionally called the *weber-turn*, even if only to indicate that one is referring to flux linkages, and not simply to fluxes, when it is used instead of the weber.)

Note 1. Consider the function $f(t) = A\sin(\frac{2\pi}{T}t + \alpha)$, where A, T, and α are constants. For all *t* we have $f(t) = f(t+T)$, and for this reason *T* is call a (the) *period* of f(t). The number ν of periods *T* per unit of *t* is called the *frequency* of the function; hence $\nu = 1/T$. Also, the quantity $\omega = \frac{2\pi}{T} = 2\pi\nu$ is called the *angular frequency*, or more simply the *2π-frequency*, of the function. The unit of f(t) is that of *A*; and α is dimensionless, of course. For ν and ω the unit is the inverse unit of *t*; hence if *t* is time, the unit of both ν and ω is the *inverse second* (sec^{-1}), in the MKSC-system. Nevertheless, in referring to frequencies we conventionally say *cycles per second* (abbreviated cyc/sec, or cps, or simply $\sim$) or Hertz, abbreviated Hz. or hz., to indicate this, and in referring to angular frequencies we conveniently say *radians per second* (rad/sec).

Problem 3. The magnetic flux through a surface is the surface integral of the magnetic induction *B* (a limit of a sum of values of *B* multiplied by portions of areas). Show that the unit of magnetic induction is the $weber/mt^2$.

Problem 4. The line integral of the magnetic field intensity H taken around a closed path (called the *magnetomotive force* of the path, and abbreviated *MMF* or *mmf*) is equal, except for a dimensionless factor (in a Giorgi System of units), to the total electric current enclosed by the path (including the *displacement current*), in accordance with the *law of Ampère*. The unit of mmf is thus the ampère, although it is usually expressed as the *ampère-turn* in order to indicate that one is referring to a mmf and not simply to a current. Show that the unit of magnetic field intensity, H, is the ampère-turn/meter.

Problem 5. From $B=\mu H$ show that the unit of *permeability*, μ, (also known as the *magnetic inductive capacity*) is the henry/meter.

Problem 6. The electric field intensity is defined as the force per unit electric charge. Show that the unit is the volt/meter.

Problem 7. In an isotropic medium the electric energy in a dielectric is the volume integral of the electric field intensity multiplied by the *dielectric displacement* (except for a dimensionless factor). Show that the unit of dielectric displacement is the coulomb/mt^2.

Problem 8. Considering that the dielectric displacement is equal to the *permittivity* κ (also known as the *dielectric constant* (especially if it is constant) or as its (electric) *inductive capacity*; see note on p. 25) multiplied by the electric field intensity, show that the unit of permittivity is the farad/meter.

Problem 9. Show that the unit of energy is equal to the unit of mass mutiplied by the unit of velocity squared.

Problem 10. Show that the unit of permeability multiplied by the unit of permittivity, in a Giorgi system of units, is the inverse unit of velocity squared.

Problem 11. In the *practical* (also called „*technical*") *system of units* known as the meter-kilogram weight-second system, a unit of length (the meter), a unit of force (the *kilogram weight*, abbreviated kgr), and a unit of time (the second), are taken as the fundamental units. The kgr is defined as the weight of a kgm (the unit of mass in the MKSC-system), so that it is a unit of force which depends upon position on the earth. In this system, the unit of mass is called the geo-kilogram, abbreviated gkg. At a place on the earth where the acceleration due to gravity is g, standardized to 9.80665, (mt/sec^2), show that:

1 kgr = g newtons and 1 gkg = g kgm.

Problem 12. The magnetic moment of a current loop, or magnetic dipole, is defined as the product of the current in the loop by the area of the loop. Show that the unit of magnetic moment is the amp.mt^2. What can you say about the unit for magnetic poles? (See Stratton's book on Electromagnetic Theory, Mc Graw-Hill Book Co., pp. 241-2.)

In the following, when no mention is made of a unit, in speaking of the magnitude of a quantity, it shall be understood that it is the corresponding MKSC-unit. Otherwise the unit should be stated.

For many practical purposes it shall be found convenient to construct auxiliary units with the following prefixes and abbreviations:

micro... = 10^{-6}... = μ...
milli... = 10^{-3}... = m...
centi... = 10^{-2}... = c...
deci.... = 10^{-1}... = d...

mega... = 10^{6}... = M...
kilo... = 10^{3}... = K... or k...
hecto.. = 10^{2}... = H...
deca... = 10... = D...

For example, μf(= μfarad) = 10^{-6} farads, and $\mu\mu f$ = 10^{-12} farads, are frequently used as units of capacitance; and the kilowatt (abbreviated Kw or kw) = 10^{3} watts is frequently used as a unit of power.

Unfortunately the abbreviations mentioned above are not used uniformly in the literature.

In practice, the very frequently used unit of resistance (ohm) is often denoted by the capital greek letter Ω, and the mho , also called a siem$\overset{e}{\text{a}}$ns, by ℧.

There is a question about rational and irrational systems of units which the reader should consult, for example, in the originator's own paper: O. Heaviside, The position of 4π in Electromagnetic Units (Nature, 1892); or in Appendix C of the book on Electric Circuits, by the E.E. Staff from M.I.T. (Wiley and Sons); or in Stratton's book mentioned in problem 12. Essentially, in a rationalized system a 4π is introduced in the denominators of Coulomb's laws to begin with, whereas in unrationalized systems this is not done. In this way the factor 4π is avoided in many formulae in rationalized systems. In the unrationzlized MKSC-system of units, the mmf taken around a closed path is taken as 4π multiplied by the enclosed current, and the energy in a dielectric is taken as half the volume integral of the electric field intensity by the dielectric displacement, divided by 4π . In rationalized systems these 4π are omitted.

We shall not go into these matters any further because the question of rationalization is irrelevant to the units of the quantities with which we shall be concerned in this book.

Note. A generic name for permittivity and permeability is „inductive capacity." The ratio of an inductive capacity (of a substance) to that of free space is called the relative inductive capacity (of the substance). Thus, the relative magnetic inductive capacity or relative permeability (of a substance of permeability μ) is μ/μ_0 , where μ_0 is the permeability of free space

(= 10^{-7} in the unrationalized, and = $4\pi\cdot 10^{-7}$ in the rationalized, MKSC-system of units), and the relative electric inductive capacity or relative permittivity (of a substance of permittivity κ) is κ/κ_0, where κ_0 is the permittivity of free space (= $1/c^2\mu_0$, where c is the speed of light in free space = $3\cdot 10^8$ mt/sec approximately). The relative permittivity is frequently called the specific (electric) inductive capacity, and some authors even reserve the name „dielectric constant" for this term.

Problem 13. In the „English System" of units, also called the foot-pound-second (fps) system, a unit of length (the foot = 1200/3937 mt), a unit of mass (the pound = 0.4535924277 kgm), and a unit of time (the second), are taken as fundamental units. (In England the (british) foot is taken = 1200/3937.0113 mt, and the (british) pound is taken = 0.45359243 kgm.) The unit of force in this system is called the poundal. In technical work, a derived unit of force called the pound-force (or pound-weight) is extensively used; it is defined as the weight (and therefore depends on position on the earth) of a pound of mass (the unit of mass in the fps-system). Alongside, a derived unit of mass, called the slug, is also used; it is defined as the mass to which a constant force of one pound-weight would give a unit acceleration of one ft/sec^2. Show that, at a place on the earth where the acceleration due to gravity is g (ft/sec^2), we have:

1 pound-weight = g poundals and 1 slug = g pounds (of mass).

(By agreement, the acceleration due to gravity has been standardized to the value of 32.1739 ft/sec^2.) In the fps-system, the unit of energy is the foot.poundal, and the unit of power is the foot.poundal /second. The foot.pound-weight (= g foot.poundals) is a derived unit of energy frequently used in technical work; and alongside, the horsepower (abbreviated HP), defined as 550 foot.pound-weight/second is used as a unit of power. Show that 1 HP = 746 watts approximately.

§9. THE SIGN OF A MUTUAL INDUCTANCE AND POLARITY MARKS.

Let us again consider the system of n coils of §3. We suppose that reference directions have been assigned to these coils so that for each coil there is determined an associated positive direction for the magnetic flux through any surface bounded by that coil. Now assume that the mutual inductance $L_{k\ell}$ between the generic coils k and ℓ is not zero. Then, if there is a clearly defined magnetic path linking these coils, it shall be easy to determine the sign of $L_{k\ell}$ by inspection. The rule is that it shall be positive if the associated positive directions for the magnetic fluxes of the two coils are in the same sense along the magnetic path, and negative if they are in opposite senses. Thus, for the coils 1 and 2 of Fig. 1, the mutual inductance is positive, and for the coils 1 (or 2) and 3 the mutual inductance is negative. The reason is that a positive increment of the current through one of the coils in its reference direction shall produce a positive increment of the magnetic flux through the other coil in its associated positive direction (and hence a positive $L_{k\ell}$, in accordance with Eqs. 7, p.7), if and only if the two coils have associated positive directions in the same sense along the magnetic path linking them.

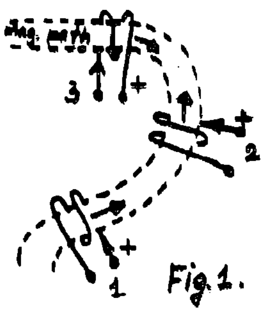


Fig.1.

A convenient method of indicating that the currents in certain directions around two coils produce magnetic fluxes in the same senses through each other is to place equal marks (like dots, asterisks, crosses, plus signs, letters, etc.) called polarity marks or, more properly, inductance or L-polarity marks, at both entrance, or at both exit, terminals. Thus, in Fig. 1, + signs could be placed as shown. They could all be placed instead at the other terminals, also.

Problem 1. Deduce the rule of signs, given above, from Lenz's law.

The use of polarity marks is very convenient for drawing purposes where winding senses are hard to indicate. Moreover, unlike the signs of the coefficients of mutual inductances, polarity marks have the advantage of being independent of the reference directions assigned to the coils; because currents entering both coils through the equally marked terminals produce magnetic fluxes in the same sense no matter how the reference directions are assigned or whether they are changed; whereas upon changing the reference direction of a single coil, the signs of all the mutual inductances of this coil with other coils change.

We may summarize the above in the following rule of signs for the mutual inductances: The mutual inductance M between two coils with assigned reference directions and given polarity marks is positive if the two reference directions are (both) directed towards the polarity marks, as in Fig. 2(a), or if both are directed away from the polarity marks, as in Fig. 2(b); and negative otherwise, as in Figs. 2(c) and (d).

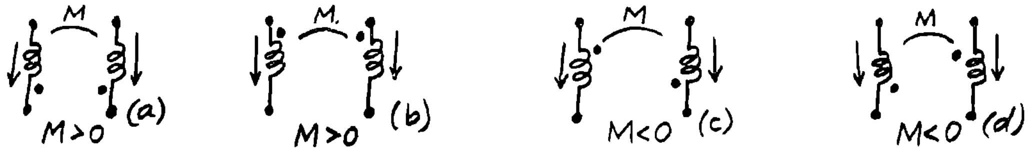


Fig. 2, exhibiting the rule of signs for mutual inductances.

The method of polarity marks is practically indispensible when the coils are concealed or when the magnetic path linking them is not clear. In this case the polarity marks can be determined by the following simple experimental method, for each pair of coils. A battery (a practical constant voltage source) is connected, through a switch, between the terminals of one of the coils and a d-c voltmeter (i.e. a directional voltmeter, an instrument that indicates the sense of a voltage rise, besides the value) is connected between the terminals of the other coil. Upon closing the inserted switch a current begins to flow from the + terminal of the battery into the former coil, with a positive rate of increase, and a momentary deflection shall be indicated on the d-c voltmeter connected to the latter. In this way the direction in which the induced voltage drops (or rises) may be observed. The polarity marks for the pair of coils are then placed at the terminals towards which the voltage rises or, in other words, at the terminals from which the voltage drops; in particular, one of the marks is placed at the terminal which is connected to the + terminal of the battery. Naturally, both polarity marks could be placed at the other terminals instead. In this way we are sure that a positive rate of change of the current in the reference direction through one of the coils shall produce an actually positive voltage drop through the other coil in the reference direction if and only if the mutual inductance (as given by the rule exhibited in Fig. 2) is positive, as can be checked by cases easily.

In carrying out this experimental method for a pair of coils belonging to a system of coils, it should be understood that all the other coils of the system are left open-circuited in the meanwhile, and that none are changed (relatively) in position.

In a given system of coils there shall be need in general of different polarity marks for each pair of coils. In some special systems, however, things can often be arranged in such a way that only one kind of polarity mark is needed and with at most only one mark being necessary for each coil. Such is the case, for example, with a system of coils all wound on a single magnetic path (e.g. a common iron core). For then there are only two possible senses, along the magnetic path, for the associated positive directions of the magnetic fluxes, and so any two directions in opposition to a third must necessarily be directed in the same sense, and so consistency in the markings necessarily results.

But this is not always the case, as the simple example of Fig. 3

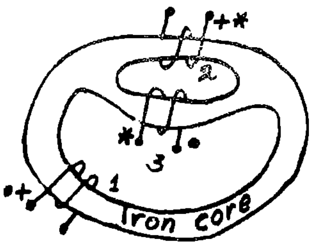


Fig. 3.

shows. The polarity marks that result from a consideration of the pairs of coils (1,2) and (1,3) do not serve for the pair (2,3). In fact, it is impossible to use a single type of polarity marks because coil 3 has marks at both terminals.

Problem 2. Consider two coils connected as shown in Fig. 4.

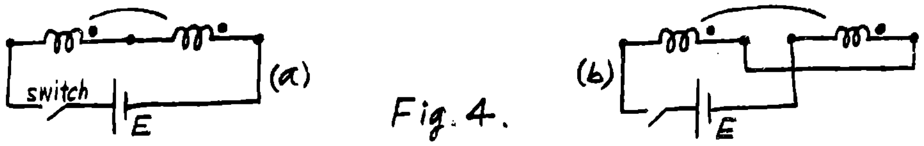


Fig. 4.

Show that for equal voltages the current in case (a) is smaller than the current in case (b). This result gives another method for the determination of polarity marks and the sign of a mutual inductance. If a voltage source of sinusoidal value $A.\sin(\omega t+\alpha)$ were used instead of a battery, show that the effective value of the current in (a) would be smaller than in (b).

Problem 3. By Lenz's law, show that for any two coils k and ℓ, the sign of $L_{k\ell}$ is equal to the sign of $L_{\ell k}$.

Let us now take up the question about the sign of a mutual invertance (or inverse mutual inductance). For this let us again consider the system of n coils of §3, each with an arbitrarily assigned reference direction (and with all initial currents = 0). Of course, if all the self and mutual inductances $L_{k\ell}$ are known (in absolute value and sign) then one may go ahead and compute the invertances $\Gamma_{k\ell}$ by Eq.(12) §3, sign and all. But just as the $|L_{k\ell}|$ are experimentally determinable, so are the $|\Gamma_{k\ell}|$; and it shall then be important to be able to determine the signs of the mutual invertances $\Gamma_{k\ell}$ directly, by experiment.

This may be done by inserting in coil k (see Fig. 5) a d-c ammeter (i.e. a directional current instrument, one that not only indicates the value of the current but also its sense) and in coil ℓ

Fig. 5

a battery and a switch, while all the other coils are short-circuited (i.e. their terminals are connected together); but no coil is to be changed (relatively) in position. When the switch is closed a positive voltage drop shall appear in coil l in the reference direction assigned to it and a momentary deflection on the ammeter (if at all) shall indicate whether the induced current in coil k increases, or not, in the reference direction assigned to it. If the induced current in coil k increases in the reference direction, the sign of Γ_{kl} is positive, and if it does not, the sign of Γ_{kl} is negative. (If there is no deflection then the invertance, Γ_{kl}, is zero.) For under the hypotheses made, all the voltage drops are zero except V_l which is positive and hence, by Eqs.(9) § 3, we will have $dI_k/dt = \Gamma_{kl} V_l$, so that, if $dI_k/dt > 0$ then $\Gamma_{kl} > 0$, and if $dI_k/dt < 0$ then $\Gamma_{kl} < 0$.

Mutual Invertance (*inverse mutual inductance*) or *Γ-polarity marks* can also be introduced. In the experimental method just explained, this is done, in pairs, by putting equal marks at the terminals where, upon closing the inserted switch, the currents enter into the coils of the pair. In Fig. 5, a mark (a + sign, say) is placed as indicated on coil l, and the same mark should be placed at the terminal a of coil k if the induced current is observed to grow in the reference direction assigned to it when the switch inserted in coil l is closed, and at the terminal b if not (if at all), while all the other coils are shorted. We shall then have the following easy *rule of signs for the mutual invertances* Γ_M:

The mutual invertance (inverse mutual inductance) Γ_M between two coils with assigned reference directions and given Γ-polarity marks is positive if the two reference directions are directed towards the

the polarity marks or if both are directed away from the polarity marks, and negative otherwise. (This rule is just like the rule of signs for a mutual inductance but, of course, the Γ-polarity marks are in general different from the L-polarity marks. Hence, Fig. 2 can also be used to exhibit the rule of signs for mutual invertances if the polarity marks shown are considered as Γ-polarity marks and if Γ_M is substituted for M.)

It should be noted that the sign of a mutual invertance between two coils depends essentially on the reference directions assigned to these coils, but not on the reference directions assigned to the other coils; however it surely depends on the presence of the other coils, unless they are not magnetically coupled with them. As a consequence, if the reference direction of a single coil is changed, the signs of all the mutual invertances of this coil with other coils also change.

Problem 4. In a system of only two magnetically coupled coils, show that the Γ-polarity marks can be obtained from the L-polarity marks by changing only one of the marks from one terminal to the other.

Problem 5. Consider a system of n coils together with their assigned reference directions and the mutual inductance polarity marks (not the Γ-polarity marks). Consider two coils with a single simple chain of magnetically coupled coils linking them (much like an ordinary chain with two ends), with a mutual invertance $\Gamma_M \neq 0$. Let p denote the number of magnetic links of the chain whose coils have their reference directions both directed towards (or both directed away from) the corresponding mutual inductance polarity marks. Show that the sign of $\Gamma_M = (-1)^p$. Why is this not a very practical rule in general? (Help yourself with a figure.)

CHAPTER II: KIRCHHOFF'S LAWS AND RELATED TOPICS.

An investigation of the currents and voltages in any interconnected collection of basic elements is founded upon two laws known as Kirchhoff's Current and Voltage Laws. The Current Law (called "Condition of Continuity" by Maxwell in his Treatise on Electricity and Magnetism) states that the (algebraic) sum of all the currents flowing away from (or towards) any junction point (a terminal, whether common or not) is zero at each instant. The Voltage Law states that the (algebraic) sum of all the voltage drops (or all the voltage rises) taken in a given sense around any closed path through elements of the collection is zero at each instant.

An electric circuit or network is defined as any collection of basic elements consistent with Kirchhoff's Laws (i.e. collections in which, for every junction point and every closed path, the corresponding Kirchhoff Law is satisfied). These shall be the only kind of collections we will consider in this book.

To solve a network will mean to find all the currents & voltages through elements of the network (for all time). In terms of these, all charges, fluxes, energies, powers, etc., can then be computed.

§1. GENERAL ELEMENTS

In practice it shall be found convenient to consider certain collections of basic elements connected in definite ways as the units of the electric circuit. These, we shall call general elements, or simply elements. One such collection is shown in Fig. 1.

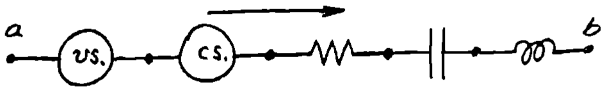


Fig. 1. The general series element.

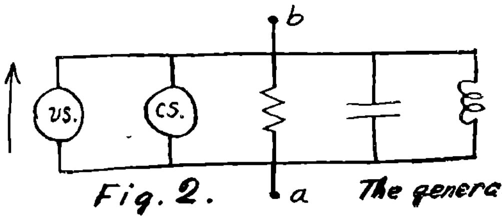


Fig. 2. The general parallel element.

Any basic element may be considered as one of these elements with the other four basic elements missing, so that this is, indeed, a generalized element. Instead of assigning arrows to each of the basic elements, a single arrow is attached to the generalized element which is to serve as a common reference direction for all the basic elements constituting it both for currents and voltages. By Kirchhoff's Current Law, applied to each of the intermediate junction points (common to two of the basic elements of the general element), we infer that the currents in all these basic elements in the common reference direction are the same and, therefore, equal to the current entering it at the end a and leaving at the end b. For this reason we call the element shown in Fig. 1 a general series element and we speak only of a current through it, entering at end a and leaving at end b. (In general, elements always with the same current through them and connected together are said to be in series.) We then ignore the intermediate junction points and speak of ends a and b as the terminals of the generalized element and allow connections with other elements only at these terminals.

The voltage drop V between the terminals in the reference direction, being the work done by the electric field in carrying a unit electric charge through the basic elements which compose it, in the reference direction (for fixed time), is equal to the sum of the voltage drops V_{vs}, V_{cs}, V_R, V_S, V_L, in the voltage source, current source, resistor, capacitor and inductor, respectively. We therefore have for the total voltage drop in the series element:

$$V = V_{vs} + V_{cs} + V_R + V_S + V_L. \tag{1}$$

When a basic element is missing in any specific case, the corresponding term in this equation is to be omitted, of course.

Another such collection is shown in Fig. 2 (above). Any basic element is one of these with the other four basic elements absent. Here too a single arrow is arbitrarily attached to the generalized element which is to serve as a common reference direction for all the basic elements composing it, both for currents and voltages. By Kirchhoff's Voltage Law, applied to each of the "interior" closed paths (namely, the paths formed by the vs and cs, the cs and resistor, the resistor and capacitor, and the capacitor and inductor) we infer that the voltage drops (or rises) in the basic elements composing the generalized element, in the common reference direction, are all the same. For this reason we speak only of *a* voltage drop (or rise) in the reference direction from one end of a common lead on one side to the end of another common lead on the other side of the element (the terminals *a* and *b* in Fig. 2), and we call this element a *general parallel element*. (In general, elements connected together and always with the same voltages are said to be *in parallel*.) The "interior" closed paths are then ignored and the structure is taken as a unit with the terminals *a* and *b*.

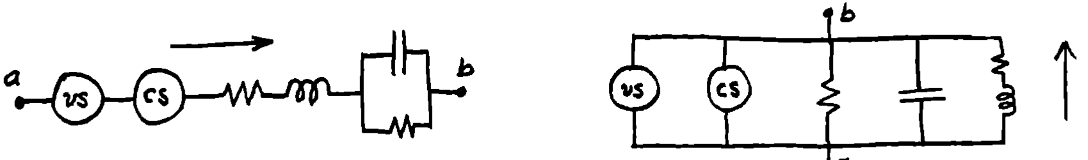
By Kirchhoff's Current Law, the current entering the element through the terminal *a* must equal the sum of the currents $\underline{I_{vs}}$, $\underline{I_{cs}}$, $\underline{I_G}$, $\underline{I_C}$, $\underline{I_\Gamma}$, in the voltage source, current source, resistor, capacitor, and inductor, respectively; and again by the Current Law, this sum must be equal to the current leaving through the terminal *b*. Therefore the current entering at one terminal is equal to the current leaving through the other, and we may speak of *a* current $\underline{I}$ through the generalized element in the reference direction given by:

$$I = I_{vs} + I_{cs} + I_G + I_C + I_\Gamma . \qquad (2)$$

If a basic element is missing in a specific instance of the typical general element, the corresponding term in this equation should be understood to be omitted, of course.

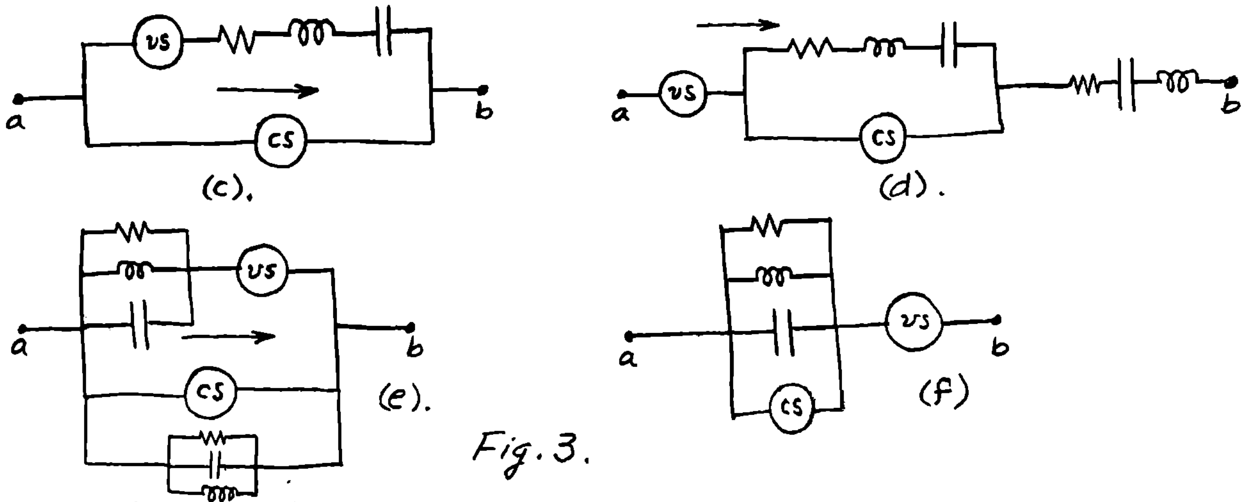
When dealing with series elements we shall usually speak of resistances, elastances and inductances of the basic passive elements; because in a series connection it is found that resistances are added together (whereas conductances are not), and the same is true of elastances (but not of capacitances) and of inductances. Similarly in dealing with parallel elements we shall usually speak of conductances, capacitances and invertances (inverse inductances); because in parallel elements it is found that these are the parameters that add up (and not the resistances, elastances and inductances). However this is just a matter of convenience.

In Fig. 3 are shown several other structures that are sometimes also convenient to consider as (typical or generic) units of a network. The struc-



(a). The mixed series element.

(b). The mixed parallel element.



(c).

(d).

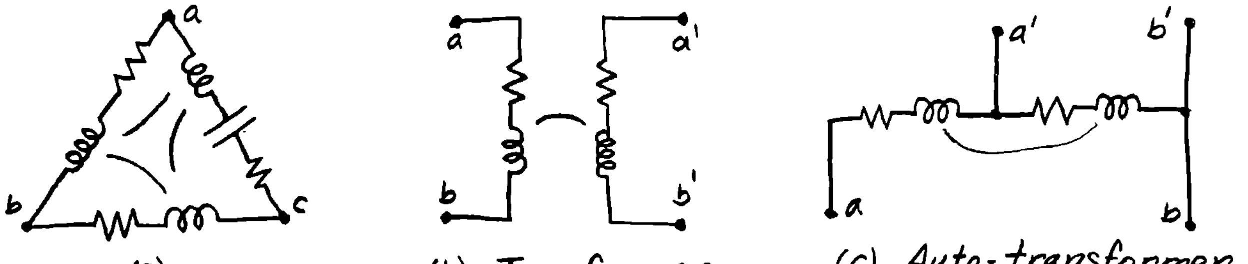
(e).

(f)

Fig. 3.

tures (a) and (b) are the more important ones. The others can be shown to be reducible (by interchanging sources according to certain results that can hardly be explained at this point) to particular cases of the general series or parallel elements. (Cf. Ch. IX, §8.

The series and parallel elements, like the basic elements, are examples of two-terminal structures (i.e. elements with only two terminals where connections can be made with other elements). *N-terminal elements* are structures (i.e. definite interconnected collections of basic elements considered as units) with *N* terminals where connections with other elements can be made. For example, the structure shown in Fig. 4(a) is a 3-terminal element, while the structures shown in Fig. 4 (b) and (c) are 4-terminal elements. The structure 4(b) is known as a (single phase) *transformer*, and the structure 4(c) is known as an *auto-transformer*.



(a). (b). Transformer. (c). Auto-transformer.

Fig. 4.

Problem 1. Try to express the voltage drops between the terminals a & b and a' & b' in terms of the currents flowing between these terminals, for the two structures shown in Fig. 4 (b) & (c). (In a general sense, any two of the four quantities: the two voltage drops and the two terminal currents, can be expressed in terms of the other two.)

§2. THE GRAPH OF A NETWORK AND ITS COMBINATORIAL STRUCTURE.

Let us assume that a certain type of two-terminal element has been chosen. In any network of such elements a node (or "junction point) is defined to be any terminal of an element (of the kind chosen) whether common or not to several elements; and a mesh is defined as any simple closed path (i.e. a closed path without "crossings") through elements of the network. By "without crossings" we mean that no element and no node is traversed twice by the path, and it has nothing to do with how a network is drawn.

To be a little more formal about the concept of a mesh, assume that all the generalized elements of the network are numbered 1,2,...,N. Then any closed path through elements of the network can be defined by a sequence of numbers: $k_1k_2\ldots k_M$ ($M\leq N$), such that consecutive numbers of the sequence belong to elements of the network with a terminal in common and such that the elements k_1 and k_M also have a terminal in common. (Of course, it is to be understood that any cyclic permutation of the above sequence defines the same closed path.) Then a mesh is defined as a closed path through elements of the network with no repeated elements and such that no three elements in the sequence defining the closed path have a terminal in common. The idea is that if we traverse the elements $k_1, k_2, \ldots, k_M$ in succession we do indeed go around a closed path in the intuitive sense; and if in so doing no element is traversed twice and no three elements traversed have a terminal in common then the closed path will have no crossings (i.e. the closed path will be a mesh).

Now assume that a line (or arc) segment (to be called an edge) is substituted for each generalized element of a given network. The result, which is an interconnected collection of edges and junction points (also called *vertices*), similar to the given network, is called the *graph* of the network. If, furthermore, we arbitrarily assign reference directions to the elements of the network, we say that the network and its elements have been oriented, or directed, and similar reference directions may (and should) be assigned to the edges of its graph. In this way a one-to-one correspondence between elements and edges and between nodes and vertices can be established which preserves orientations and the relation of incidence (i.e. of being connected to). For this reason the network and its graph are said to have the same *combinatorial structure* and to be *isomorphic* (in this sense). One then takes the liberty of using the terminology of a network and of its graph indistinctly, and many a time a reference to a network is made by a reference to its graph (in order to save time, mostly).

By agreeing, as is often done in practice, on understanding that all terminals with the same mark are really connected into a common node, i.e. *identified*, one need not actually draw a graph always in the strict sense of the word but only a collection of line (or arc) segments with the appropriate identification of the terminals (i.e. with the proper terminal marks). No restriction is necessary on the kind of terminal marks to be used as long as they are used unambiguously. Of course, the same technique may be used for a network also; thus a network may be completely specified by giving the details of the separate elements with the proper identification of the terminals. (A commonly used mark is: ⏚ ; in general this mark is used for a common solid connection to earth so that it is usually referred to as ground; but this need not always be the case.)

A word of caution is at place in connection with the convention on the identification of terminal marks. In a network in which the convention is used one must always be careful in connection with Kirchhoff's Current Law to include the currents in all the elements with the same terminal marks.

The combinatorial structure of an oriented network or graph is defined formally by certain functions of two variables whose values are called incidence numbers. In order to define these, let us number all the nodes, the elements (of the kind chosen), and all the meshes of the network. Let us further assume that each mesh has been oriented (or directed, i.e. that a reference direction has arbitrarily been assigned to each mesh); this may be done in a convenient way by assigning to each mesh: $k_1k_2\ldots k_M$, the sense in which this mesh is described in going through the elements of this sequence in the order given (or in any cyclic permutation thereof), if the sequence has at least three members, or by assigning to the mesh the sense defined by the reference direction of the first element in a specific sequence defining the mesh, for example.

The incidence number of element k with the node n, which we shall denote by (k,n) (omitting the functional letter, as in $F(k;n)$), is defined as follows:

$(k,n) = +1$, if and only if the element $\underline{k}$ has exactly one terminal at the node $\underline{n}$ and is directed away from it;

$(k,n) = -1$, if and only if the element $\underline{k}$ has exactly one terminal at the node $\underline{n}$ and is directed towards it;

$(k,n) = 0$, otherwise, i.e. if the element $\underline{k}$ has both terminals, or none, at the node $\underline{n}$.

Note that the first argument in the function (k,n) is the number of an element and the second argument is the number of a node.

When $(k,n) \neq 0$ for some particular $\underline{k}$ and $\underline{n}$, i.e. when $(k,n)=\pm 1$, we shall say that the element $\underline{k}$ incides on the node $\underline{n}$, or that the element $\underline{k}$ and the node $\underline{n}$ are incident.

The incidence number of an element $\underline{k}$ with a mesh $\underline{m}$ (defined by a sequence $k_1k_2 \ldots k_M$ of elements), which we shall denote by $[k,m]$, is defined as follows:

$[k,m] = +1$, if and only if the element $\underline{k}$ belongs to the mesh $\underline{m}$ (i.e. the element $\underline{k}$ is one of the elements in the sequence of elements $k_1k_2 \ldots k_M$ defining the mesh $\underline{m}$) and the mesh $\underline{m}$ is directed through the element $\underline{k}$ in the same sense as the element $\underline{k}$;

$[k,m] = -1$, if and only if the element $\underline{k}$ belongs to the mesh $\underline{m}$ and the mesh $\underline{m}$ is directed through the element $\underline{k}$ in the sense opposite to that of the element $\underline{k}$;

$[k,m] = 0$, otherwise, that is, if and only if the element $\underline{k}$ does not belong to the mesh $\underline{m}$.

Note that the first argument of the function $[k,m]$ is the number of an element while the second argument is the number of a mesh. Note also that we are here giving a mesh the character of a face (which may be considered as a generic surface bounded by the mesh), as shall become even clearer later on.

When $[k,m] \neq 0$, i.e. when $[k,m] = \pm 1$, for some particular $\underline{k}$ and $\underline{m}$, i.e. when the element $\underline{k}$ belongs to the mesh $\underline{m}$, we (also) say that the element $\underline{k}$ incides on (or with) the mesh $\underline{m}$ or that the element $\underline{k}$ and the mesh $\underline{m}$ are incident.

We shall now prove the following important theorem.

In any network the sum of the products $(k,n)[k,m]$ taken over all elements $\underline{k}$ is always zero, i.e.

$$\sum_k (k,n)[k,m] = 0 \tag{1}$$

for all nodes $\underline{n}$ and all meshes $\underline{m}$.

The proof is as follows:

1°). If the node $\underline{n}$ is not on the mesh $\underline{m}$ (i.e. if the node $\underline{n}$ is not the terminal of an element belonging to the mesh $\underline{m}$) then $(k,n) = 0$ for all $\underline{k}$ for which $[k,m]$ is not 0. Therefore all the terms of the sum in eq. (1) are zero and consequently the equation holds.

2°). If the node $\underline{n}$ is on the mesh $\underline{m}$ and this mesh has a single element $\underline{k_1}$ (say) then this element must have its two terminals on the node $\underline{n}$, so that $(k_1,n) = 0$. Furthermore, for all $k \neq k_1$ we have $[k,m] = 0$, because no other element belongs to the mesh $\underline{m}$. Consequently all the terms in eq. (1) are 0 and the result holds.

3°). Finally, if the node $\underline{n}$ is on the mesh $\underline{m}$ and this mesh has more than one element then the node $\underline{n}$ shall be the common terminal of exactly two elements of the mesh $\underline{m}$, the elements $\underline{k_1}$ and $\underline{k_2}$ (say), and we shall have $(k,n) \neq 0$ only for $k=k_1$ and $k=k_2$, so that the sum in eq. (1) reduces to: $(k_1,n)[k_1,m] + (k_2,n)[k_2,m]$. Now if $\underline{k_1}$ and $\underline{k_2}$ are both directed towards, or both are directed away from, the node $\underline{n}$ then $(k_1,n) = (k_2,n)$ while one of: $[k_1,m]$ and $[k_2,m]$ is equal to +1 and the other is equal to −1, so that the sum is zero.

On the other hand, if one of the elements k_1, k_2 is directed towards the node n and the other is directed away from the node n then $(k_1,n) = -(k_2,n)$ while $[k_1,m] = [k_2,m]$ so that the sum is again zero.

Thus we see that eq. (1) always holds and the proof is complete.

In practice, when a mesh is specified by a sequence of elements composing it, it shall be found convenient to complete the information by indicating which elements are traversed by the mesh in the reference directions assigned to them and which in the opposite sense. This may be done by placing plus and minus signs (respectively) above the numbers of the elements in the sequence defining the mesh (and actually the plus signs could be omitted while the minus signs are retained, as indicative of the situation). Thus if the element k_l in the mesh m given by the sequence $k_1 \ldots k_l \ldots k_M$ is directed in the same sense as the mesh we would leave k_l as it stands, but if it were directed in the opposite sense we would indicate this by placing a minus sign above k_l, so that as far as k_l is concerned the mesh would be specified as: $k_1 \ldots \bar{k}_l \ldots k_M$. In this way, once the mesh is given completely in the above sense, the incidence numbers $[k,m]$ can be called out without further reference to the network under discussion. For example, if a mesh m were given as: $\bar{1}2\bar{5}$, we would infer that $[1,m] = -1$, $[2,m] = +1$, $[5,m] = -1$, while $[3,m] = 0$ (as well as $[k,m]$ for all $k \neq 1, 2, 5$) because the element 3 does not belong to the mesh m (as well as all other elements k distinct from the elements 1, 2, and 5).

§3. KIRCHHOFF'S CURRENT LAW.

The basis of Kirchhoff's Current Law is the fact that electric charge cannot accumulate at a node, which means that the (algebraic) sum of all the instantaneous currents entering (or leaving) a node must always be zero. Actually all this goes back to Maxwell's Electromagnetic Theory, as a direct consequence of the (for all practical purposes) divergenceless character of electric currents in homogeneous isotropic conductors. Of course, in Electric Circuit Theory one assumes idealized nodes (in which one postulates that no charge can accumulate) and just up to what point one can assume an actual junction point to be an idealized node depends on just how small the displacement current vector in a conductor is. In any case, ~~an~~ actual junction points may very well be represented by ~~an~~ idealized nodes and a system of condensers, so that one is again led back to the concept of an idealized node.

Problem 1. The density ρ of electric charge in a homogeneous isotropic conductor of permittivity κ and conductivity γ can be shown to satisfy the equation: $\kappa\frac{\partial\rho}{\partial t}+\gamma\rho=0$. Show that at each point at which $\rho=0$ at any one instant is always $=0$.

Problem 2. Let I_k be the electric current through the element k in the reference direction assigned to it. If this element k has only one terminal at a node n and is directed away from the node then the current leaving the node n through the element k is I_k; but if the element k is directed towards the node n then the current leaving the node n through the element k is $-I_k$; while in any other case the current leaving the node n through the element k is zero (when the element k has both its terminals or no terminal at the node n). Show that in every case the current leaving the node n through the element k is always given by: $(k,n)I_k$.

Let us now consider an oriented network with all its nodes and elements numbered. Kirchhoff's Current Law for any node n can be conveniently expressed analytically with the aid of the incidence numbers (k,n) between the elements k and the node n. For this purpose let I_k denote the current through the element k in the reference direction arbitrarily assigned to it. Then $(k,n)I_k$ is clearly the current leaving node n through the element k and therefore the sum of all the currents leaving the node n through all the elements is: $\sum_k (k,n)I_k$ taken over *all* elements. By Kirchhoff's Current Law this must be zero; hence the Current Law for the node n can be expressed analytically as follows:

$$\sum_k (k,n)I_k = 0. \tag{1}$$

We will now define a few terms to be used later on. We shall say that two distinct nodes n' and n'' of a network are *mutually accessible* (or that one is accessible from the other) if and only if there is (a path joining them defined by) a sequence of elements: $k_1 k_2 \dots k_M$ such that consecutive elements of the sequence have a node in common and such that the node n' is a terminal of the element k_1 and the node n'' is a terminal of the element k_M. A set of elements is said to be *connected* if and only if all the terminals of the elements of the set are accessible from a terminal of one element of the set. A non-empty connected set of elements of a network is called a *part of the network*. Two parts of a network are said to be *disjoint* or *detached* (or *separate*) if and only if no node of one part is accessible from a node of the other part. A part of a network which is detached from all other parts (not included in the given part), i.e. a maximal part, is called a *component* (or a *detached* or *separate* (connected) *part*) of the network. An element of a network is called *inessential* if and only if it has a loose terminal or if it has its two terminals at two distinct nodes of a component which splits into two components upon removal of the element in question; otherwise it is called and *essential element*.

Problem 3. Show that an inessential element of a network cannot belong to a mesh of the network. Also show that there always is a mesh containing any given essential element.

Problem 4. If two nodes of a network are mutually accessible, show (by deleting the parts between crossings of a path joining them) that they can be joined by a path without crossings (i.e. by a simple path). (See §2. p. 37, for the meaning of "crossings.")

Assume now that Kirchhoff's Current Law (eq. 1) is established for all the nodes n of a given component of a network. Considering all currents as indeterminates (i.e. as arbitrary variables) we shall show that exactly one of these equations is superfluous for each component (and may thus be omitted from consideration). Thence Kirchhoff's Current Law need only be established for all nodes of a network except one in each component and we shall be sure that the corresponding equations for the omitted nodes hold implicitly.

Without loss of generality we may assume the nodes of the given component to be the nodes 1, 2, ..., N. Let us denote the left member of eq. (1) by K_n. (Then Kirchhoff's Current Law for the node n would be expressed: $K_n = 0$.) We shall then have:

$$\sum_{n=1}^{N} K_n = \sum_{n=1}^{N}\sum_{k}(k,n)I_k = \sum_{k}\Big(\sum_{n=1}^{N}(k,n)\Big)I_k = 0, \qquad (2)$$

since for all elements k we have:

$$\sum_{n=1}^{N}(k,n) = 0; \qquad (3)$$

because 1°) if the element k has both its terminals at the same node then all the terms $(k,n)=0$, and 2°) if the element k has its terminals at two distinct nodes n', n'' (say) then all the terms (k,n) are zero except (k,n') and (k,n''), one of which is +1 and the other −1 whose sum is zero, and so eq. (3) (and hence eq.(2)) holds for all k.

Consequently we have: $K_1+K_2+\ldots+K_N = 0$, so that if any $(N-1)$ terms in the first member are simultaneously zero then the remaining term is also zero. (Physically this means that if no charge can accumulate at the nodes of a component of a network with the possible exception of one node then no charge can accumulate at this node either.)

Next we must show that Kirchhoff's Current Law for more than one node of a given component of a network cannot be deduced from the corresponding equations for the other nodes of the component.

To do this let us assume the contrary, i.e., let us assume that there are two nodes n' and n'' (say) of the given component such that the corresponding Kirchhoff's Current Laws: $K_{n'} = 0$ and $K_{n''} = 0$ follow from the others: $K_n = 0$, for the other nodes of the given component. Then since the nodes n' and n'' are on the same component there is a sequence of elements $k_1 k_2 \ldots k_M$ (say) of this component such that consecutive elements of the sequence have a terminal in common and such that the node n' is a terminal of the element k_1 and the node n'' is a terminal of the element k_M (in short, there is a path joining the nodes n' and n''). Without loss of generality we may assume all the elements of this sequence, $k_1 k_2 \ldots k_M$, to be oriented along the path through them in the same sense, from n' to n'' say (Fig. 1). If we now add a quantity $A \neq 0$ to all the I_k of the elements of this sequence (i.e. for the $k = k_1, k_2, \ldots, k_M$ only) then none of the K_n corresponding to nodes other than n' and n'' are altered, whereas $K_{n'}$ and $K_{n''}$ become A and $-A$, respectively, which are $\neq 0$. Thus it can not follow that $K_{n'} = 0$ and $K_{n''} = 0$ from $K_n = 0$ for all $n \neq n'$ and n''.

Fig. 1.

(Prof. Hermann Weyl has given another proof of this result by showing that a single linear relationship exists between the K_n corresponding to the nodes of a given component of a network; cf. Revista Matemática Hispano-Americana, Tomo V, Madrid, 1923.)

On account of the result just established one usually says that the *number of independent nodes* in each component of a given network is one less than the number of nodes in that component, and that the *total number of independent nodes* in the network is the number of nodes, n_n (say), minus the number of components, n_c (say).

By this we mean that the number of linearly independent Current Law equations of the network is precisely $n_n - n_c$ *which we shall denote systematically by* n'_n. *But it must clearly be understood that not any* n'_n *Current Law equations are linearly independent. One must include all the equations corresponding to all the nodes except exactly one node in each component and not, e.g., leave out the equations corresponding to two nodes from one component and include those of all the nodes of another component.*

In the following, when a network is given, we shall understand that exactly (any) one node is to be omitted from each component when Kirchhoff's Current Laws are established for the network. If the network has n_e *elements,* n_n *nodes and* n_c *components, let us number the elements consecutively from 1 to* n_e *and, upon omitting exactly one node in each component, let us number the rest of the nodes consecutively from 1 to* $n'_n = n_n - n_c$. *Then a complete set of independent Current Law equations may be written thus:*

$$\sum_{k=1}^{n_e} (k,n) I_k = 0, \qquad \text{for } n = 1,2,\ldots,n'_n (= n_n - n_c). \qquad (4)$$

(The n_c *nodes omitted in the enumeration, one in each component or separate part of the given network, shall be denoted by* O_a, *with* $a = 1,2,\ldots,n_c$, *and we shall call* O_a *the* <u>*base*</u> <u>*node*</u> *of the part* <u>*a*</u>.*)*

Kirchhoff's Current Law (eq. 1) can be used to establish the following important theorem.

The (algebraic) sum of all the currents leaving (or entering) a given (not necessarily connected) portion or section of a network is always zero.

<u>*Proof:*</u> *A portion or section of a network is specified by giving the set* <u>*S*</u> *of its elements. All the terminals of elements of* <u>*S*</u> *shall be called the nodes of the portion or section. All elements not belonging to* <u>*S*</u> *but with terminals on nodes of the section*

may be divided into three classes: K= the class of elements not belonging to S but with exactly one terminal on a node of the section, K'= the class of elements not belonging to S but with two terminals on the same node of the section, and K''= the class of elements not belonging to S but with two terminals on different nodes of S.

The currents leaving (or entering) the given section do so through elements of K, K', or K''; but since a current entering through an element of K' or K'' also leaves through it, the total current leaving (or entering) the given section shall reduce to the sum of all the currents leaving (or entering) through the elements of K only, i.e., only through the elements with exactly one terminal at a node of the section S, but which do not belong to S.

Now, by eq. (1), we have:

$$\sum_{k}(k,n)I_k = 0 \text{ (taken over all elements } k)$$

for every node n, and in particular for every node n' of the section.

Consequently, summing over all the nodes n' of the section, we get:

$$\sum_{n'}\sum_{k}(k,n')I_k = \sum_{k}\left(\sum_{n'}(k,n')I_k\right) = 0. \qquad (5)$$

Now, if k is not connected to a node of the section S, i.e., if the element k does not have a terminal at a node of the section, we shall have $(k,n')=0$ for every node n' of the section. Hence these elements may be left out in eq. (5). Moreover, if k is (the number of) an element of the class K' then $(k,n')=0$, because these elements have both their terminals at the same node; therefore these too may be left out in eq. (5). Furthermore, for all the elements k of the section S or of the class K'' the sum $\sum_{n'}(k,n')$ taken over all the nodes n' of the section is zero; because for any such element k all the terms (k,n') are zero except possibly for exactly two of the nodes n' of the section, in which case for one of them it is +1 and for the other it is -1 so that the sum is zero.

Therefore all the elements of the section $\underline{S}$ *and of the class* $\underline{K''}$ *may also be left out in eq. (5).*

In this way eq. (5) reduces to the following equation:

$$\sum_{k}{}' \sum_{n'} (k,n')I_k = 0,$$

where $\sum_{k}{}'$ *indicates a summation over all the elements of the class* $\underline{K}$ *only. Furthermore, for each element* $\underline{k}$ *of the class* $\underline{K}$ *there is exactly one node* n'_k *of the section for which* $(k,n'_k) \neq 0$, *so that the above equation immediately reduces to the following equation:*

$$\sum_{k}{}' (k,n'_k)I_k = 0. \qquad (6)$$

Now, $(k,n'_k)I_k$ *is precisely the current leaving the node* n'_k *of the section* $\underline{S}$ *through the element* $\underline{k}$ *of the class* $\underline{K}$ *(see Prob. 2, on p.43); hence the left member of the above equation (6) is the sum of all the currents leaving the given section* $\underline{S}$, *which is thus zero. Changing signs in both members of eq. (6) yields the same result for the sum of all the currents entering the section* $\underline{S}$; *and this completes the proof of the theorem.*

On account of this result we see, as is often stated, that electric current in a network behaves like the flow of an incompressible fluid. (In an incompressible fluid the amount of fluid going into a given region of space through its boundary is always equal to the amount going out.) All this can, of course, actually be traced back to the assumed divergenceless character of electric current in a network, as was stated at the beginning of this section. In fact, from a physical standpoint, the above theorem is a rather immediate consequence of this. For if electric charge is "incompressible" then it cannot accumulate in any portion of a network and consequently as much charge going in must go out, which means that the (algebraic) sum of the currents leaving (or entering) the portion must be zero always.

Several results of practical importance follow immediately from the preceding theorem:

Corollary 1. If two sections of a network are connected by n conductors (see Fig. 2) then the (algebraic) sum of the n currents through the conductors, taken in the same sense from one of the sections to the other, is always zero:

$$I_1+I_2+\ldots+I_n = 0. \qquad (7)$$

This result can also be stated thus: The sum of the currents in any of the (n-1) conductors is equal to the return current in the other.

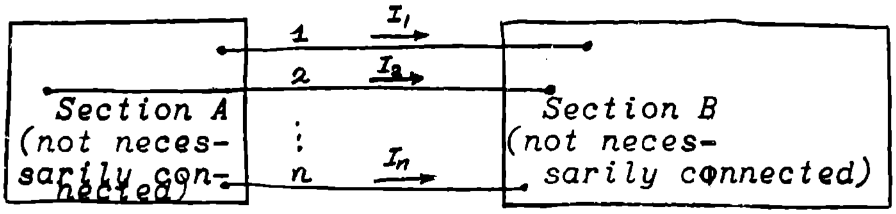


Fig. 2.

In particular, for n=1, we see that when two sections of a network are connected by a single element, the current through it is zero. Thus we infer that the current in every inessential element is zero. For n=2, we see that when two sections of a network are connected by exactly two conductors, the current in one sense through one of the conductors is equal to the current in the opposite sense through the other (see Fig. 3). For the network shown in Fig. 4 we have: $I_1+I_2+I_3 = 0$; and for the network shown in Fig. 5 we have: $I_0 = I_a+ I_b + I_c$.

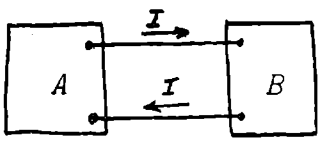


Fig.3.

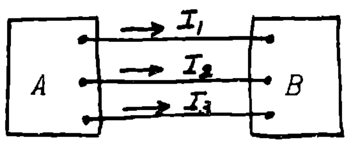


Fig.4.

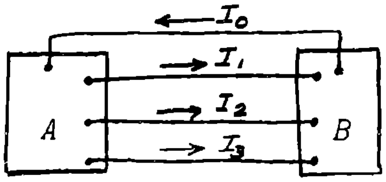


Fig.5.

To conclude this section on Kirchhoff's Current Law we may as well add that when several elements are connected in series, with a graph as shown in Fig. 6, the whole series may be considered as a unit (called a branch) if we denote all the currents through the elements of the branch with a single symbol, as is justified by Kirchhoff's Current Law applied to the intermediate nodes of the branch.

Fig.6. The graph of elements connected in series.

§4. KIRCHHOFF'S VOLTAGE LAW.

In a sense, Kirchhoff's Voltage Law is form of the Principle of Conservation of Energy: The total work done by the electric field on a unit charge in taking it around any closed path, for fixed time, is zero at each instant. Now the work done by the electric field on a unit charge in taking it through a two-terminal element, in a given direction, is the (algebraic) voltage drop between its terminals in the given direction, and the total work done in taking it around any closed path, in a given direction, through elements of a network is the (algebraic) sum of the voltage drops in the traversed elements in the given direction. This must be zero according to the statement given above; and this is precisely the statement of the Voltage Law.

As a matter of fact, however, the situation is somewhat more complicated than it seems, principally due to the idealized character given to the basic elements, especially the coils. In reality, then, one actually postulates the Voltage Law of Kirchhoff.

In order to appreciate just how exact this is and at the same time to see what limitations there may be in practical applications, one can go back to Maxwell's Electromagnetic Theory, and more specifically to Faraday's Induction Law (which states that the line integral of the electric field intensity, i.e. the work done by the electric field on a unit charge, taken around a closed path for fixed time, is equal to the negative (by Lenz's Law) time rate of change of the magnetic flux linking with the closed path in the associated positive direction). If the closed path is taken to be any oriented mesh of a network, the line integral taken around the mesh can be expressed as the sum of the line integrals taken through the elements of the mesh, due to the additive property of the integral.

The line integrals taken through voltage sources, current sources, resistances, and condensers, give the voltage drops in these elements, while the line integral taken through a practical coil is the voltage drop in the resistance of the coil which, due to the idealization, is considered apart. On the other hand, the flux linking with the mesh is the flux ϕ(say) through the face bounded by it and is thus practically equal to the (algebraic) sum of the flux linkages Ψ_k (say) of all the coils in the mesh (all taken in the positive direction associated to the direction arbitrarily assigned to the mesh). Therefore the time rate of change of ϕ is practically equal to the sum of the derivatives of Ψ_k. The latter (cf. Ch. I, §3) are the voltage drops in the idealized coils of the mesh, which together with the voltage drops previously obtained from the line integrals must add (algebraically) up to zero; as can readily be seen from Faraday's Induction Law: the line integral $= -\frac{d\phi}{dt}$, when it is expressed in the form: the line integral + $\frac{d\phi}{dt} = 0$. This is the Voltage Law with the limitation that the time derivative of the flux through the face bounded by the mesh must not differ appreciably from the time derivative of the sum of the flux linkages of the coils in the mesh. In any case, an appropriate ideal coil may be inserted in each mesh to take account of the difference, and in this way we are again led back to the Voltage Law as stated before.

As a consequence of the validity of Kirchhoff's Voltage Law for all closed paths of a network there shall be a definite voltage drop between any two given nodes when they are in the same component, i.e. when they are mutually accessible. Because, then, there is a sequence of connected elements (a path P_1) joining them and the appropriate sum of the voltage drops in these elements is *a* voltage drop between the given nodes and, moreover, *any* other path P_2 joining them together with the former path P_1 form a closed path (see Fig. 1) on p.54 for which Kirchhoff's Voltage Law assures us that the voltage drop V' (say) along P_1 from the node n_1 to the node n_2 plus the voltage drop V'' (say) along the path P_2 from n_2 to n_1 is zero, so that $V' = -V''$ which is the voltage drop along P_2 from n_1 to n_2. Thus the two voltage drops are the same and, consequently, there is a unique voltage drop from the node n_1 to the node n_2, the same for all paths (through elements of the network) joining them.

On account of this, one can always speak of *the* voltage drop (or *the* voltage rise) from each node to any *one* particular node in the same component. In the following we shall usually take this particular node in a component to be the arbitrarily chosen base node in that component (cf. §3, p.47), and we shall then speak of the *potential* U_n (say) of the generic node *n* (with respect to the base node chosen in the same component) to mean the voltage drop from the node *n* to the base node in the same component. We shall then have, in particular, for the base (or reference) nodes 0_a ($a=1,2,\dots,n_c$) that $U_{0_a} = 0$. Any voltage drop or rise, then, may also be considered as a difference of potentials, or *potential difference*, as is usually said.

Problem 1. Show that the voltage drop $V_{n'n''}$ (say, to use what is known as the *double-subscript notation for voltage drops*) from the node *n'* to the node *n''* in the same component of a network is equal to the difference: $U_{n'} - U_{n''}$. The voltage rise $E_{n'n''}$ (say, using what is known as the *double-subscript notation for voltage rises*) from *n'* to *n''* (i.e. the voltage drop $V_{n''n'}$ from *n''* to *n'*) is then: $U_{n''} - U_{n'} = -V_{n'n''}$. In particular, when *n'* and *n''* are the terminals of the same element *k* (say), show that the voltage drop on *k* (in a reference direction arbitrarily assigned to it) is equal to the sum $\sum_n (k,n)U_n$ taken over all the nodes of the network (and also that it is equal to the sum $\sum_{n=1}^{n_n'} (k,n)U_n$ taken over a given complete independent set of nodes $1,2,\ldots,n_n'$ in the network).

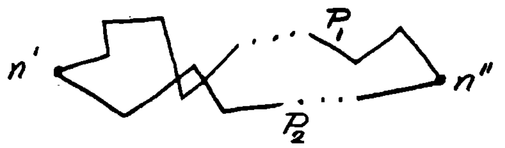


Fig. 1.

Let us now consider an oriented network with n_e two-terminal elements numbered consecutively from 1 to n_e. Kirchhoff's Voltage Law for any oriented mesh m can be conveniently expressed algebraically with the aid of the incidence numbers $[k,m]$ between the elements k and the mesh m (cf. §2). If V_k denotes the voltage drop in the element k in the reference direction assigned to it then clearly $[k,m]\cdot V_k$ is the voltage drop in this element k in the reference direction assigned arbitrarily to the mesh m, when k is an element of the mesh m, and is zero when k is not an element of the mesh m. Therefore the sum $\sum_k [k,m]\, V_k$ taken over all elements k of the network is the (algebraic) sum of the voltage drops taken around the mesh m in its assigned reference direction. The Voltage Law for the generic mesh m may therefore be expressed as follows:

$$\sum_{k=1}^{n_e} [k,m]\, V_k = 0. \qquad (1)$$

Assume now that these equations (1) are established for all possible meshes m_1, m_2, ... of the network. Any set $\{m_a\}$ of meshes shall be called a linearly independent set of meshes if and only if the corresponding linear combinations:

$$f_a = \sum_{k=1}^{n_e} [k,m_a]\, V_k, \qquad (2)$$

(considering the V_k as indeterminates or arbitrary variables) are linearly independent. If these linear forms are linearly dependant then the corresponding meshes are called linearly dependant.

Kirchhoff's Voltage Law for the mesh m_a may be expressed: $f_a = 0$, and we shall take a statement about the linear independence of a set of equations of the form $f_a = 0$ to be equivalent to a corresponding statement about the linear independence of the corresponding forms f_a (and therefore also of the corresponding meshes). We shall presently show that in each component, or separate part, of the network there shall only be a need of establishing Kirchhoff's Voltage Law for a certain number of independent meshes equal to the number of elements minus the number of independent nodes in that component, in the sense that these meshes form a complete independent set of meshes for that component, meaning that if Kirchhoff's Voltage Law is satisfied for the meshes of this complete independent set then Kirchhoff's Voltage Law for every (not necessarily simple) closed path through elements of that component is also satisfied.

Concerning the important concept of linear dependency we wish only to mention the following results. (The reader may further consult standard texts, e.g., M. Bôcher, Higher Algebra (Macmillan) Ch. III & IV; G. Birkhoff & S. MacLane, Modern Algebra (Macmillan) Ch. VII, §§4-6; P. Halmos, Vector Spaces (Princeton) §§4-6; or the charming book by van der Waerden, Moderne Algebra (Springer, Berlin) 933, of which there is now an English translation edited by F. Ungar Publishing Co., New York.) See also the appendix to chapter VI of this book, pp. 185-190, for a practical procedure of analysing linear dependence.)

19. **A** *linear form f (or a linear combination, i.e., a homogeneous polynomial of the first degree) in the variables V_k is said to be linearly dependent on the (finite) set of linear forms f_a $(a=1,2,\ldots,n)$ of the V_k if and only if there are numbers A_a such that $f = \sum_{a=1}^{n} A_a f_a$; or what is the same thing, if and only if there is a linear relation of the form: $Af + A_1 f_1 + A_2 f_2 + \ldots + A_n f_n = 0$, with $A \neq 0$. Accordingly, the "null" form $f=0$ is linearly dependent on every set of forms; and we will also say (for the sake of completeness) that it is linearly dependent on the empty set (containing no forms). It can then be shown that if f is linearly dependent on*

the forms f_1, f_2, ... , f_n but not on f_1, f_2, ... , f_{n-1} then f_n is linearly dependent on f, f_1, f_2, ... , f_{n-1}; and also that if $\underline{f}$ is linearly dependent on the set of forms, $\{f_a\}$, then it is also linearly dependent on any set of forms including the set $\{f_a\}$.

2°). The linear forms f_a $(a=1,2,\ldots,n)$ are said to be <u>linearly independent</u> if and only if no one of the f_a is linearly dependent on the others. It is then easy to show that if the forms f_a are linearly independent then (and only then) $\sum_{a=1}^{n} A_a f_a = 0$ implies that all the $A_a=0$. In particular, a (set with a) single form $\underline{f} \neq 0$ is linearly independent; and (for the sake of completeness) the empty set (with no forms) is always considered to be a linearly independent set. Also, if $f_1,\ldots,f_{n-1}$ are linearly independent but not $f_1,\ldots,f_n$ then f_n is linearly dependent on the forms $f_1,\ldots,f_{n-1}$.

3°). It can be shown that every finite set (with a finite number) of forms f_a of the (arbitrary) variables V_k $(k=1,2,\ldots,n_e)$ always has a linearly independent (but not necessarily a proper) subset of forms on which <u>all</u> the forms f_a are linearly dependent. Such a linearly independent (maximal) subset is called a <u>basis</u> for (or of) the given set of forms.

4°). Two finite sets of linear forms $\{f_a\}$ and $\{g_b\}$ (not necessarily linearly independent) are called <u>equivalent sets</u> if and only if each form f_a is linearly dependent on the forms g_b and, conversely each form g_b is linearly dependent on the forms f_a. It can then be shown that two finite equivalent sets of linearly independent linear forms must each have the same number of elements (forms); also, all bases of a given finite set of linear forms must have the same number of elements (i.e. the number of elements in a basis is invariant under a change of basis).

<u>Problem 2</u>. Consider an arbitrary oriented network with n_e elements. Consider the totality of Kirchhoff's Voltage Law equa-

tions: $f_m=\sum_{k=1}^{n_e}[k,m]V_k=0$ for all the meshes of the network. Show that it is always possible to choose a basis for the totality of linear forms f_m such that any particular essential V_k (i.e. a V_k corresponding to an essential element) appears (with a non-zero coefficient) in exactly one of the linear forms of the basis.

Show also that the linear forms corresponding to a complete independent set of meshes form a basis for the forms f_m, and viceversa.

We shall now show by mathematical induction on the number n_e of elements in a component (i.e. a detached connected part) of a network that the number of meshes n_m in a complete independent set of meshes is equal to the number of elements plus one minus the number n_n of nodes in the component, that is:

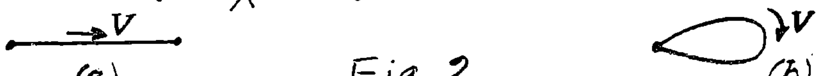
$$n_m = n_e + 1 - n_n = n_e - (n_n - 1) = n_e - n_n', \qquad (3)$$

where $n_n'=n_n-1$ is the number of independent nodes in the component.

(By the results mentioned above (the 4°) it is clear that all complete independent sets of meshes of a given component (and also) of a network have the same number of meshes, and so there is a definite number that can be called <u>the number of independent meshes of the component</u> (and also) <u>of the network</u>.)

The proof of the theorem mentioned before is as follows.

1°). If the component has a single element, it must either have its two terminals loose (as in Fig. 2a) or connected together (as in Fig. 2b). In the first case there is no mesh in the component and Kirchhoff's Voltage Law for any closed path through the element of the component is trivially satisfied. In the second case there is (essentially) a single mesh for which Kirchhoff's Voltage Law is V(=the voltage drop in the element) $=0$, and so Kirchhoff's Voltage Law for any closed path (perhaps going around several times) through the element is also trivially valid.



(a) (b)

Fig. 2.

2°). Now assume the relation (3) to hold for <u>any</u> component <u>C</u> with <u>n</u> elements and let a new element be added by connecting it to <u>C</u> to form the new component C' with (n+1) elements. This may be done in only three ways: (1°) by connecting only one of the terminals of the new element to one of the nodes of <u>C</u> and leaving its other terminal loose (Fig. 3a), (2°) by connecting the two terminals of the new element together to one of the nodes of C (Fig. 3b), and (3°) by connecting the two terminals of the new element to two distinct nodes of <u>C</u> (Fig. 3c).

In the first case (1°) the new element is an inessential element and no new mesh (and essentially no new closed path) is introduced into the component C' so that n_m is unchanged; but the number of nodes has increased by one and this cancels the unit increment of the number of elements in eq. (3) which then remains valid.

In the second case (2°) exactly one new mesh is introduced into the component (which is clearly independent of the meshes in C and) for which Kirchhoff's Voltage Law is V(=the voltage drop in the new element)=0, while the number of nodes is unchanged. Moreover, Kirchhoff's Voltage Law for any closed path in C' can be reduced to Kirchhoff's Voltage Law for a closed path in C (by deleting the new element from the path and) by placing V=0; hence Kirchhoff's Voltage Law for any closed path in C' can be obtained from V=0 and Kirchhoff's Voltage Laws for any complete independent set of meshes for C. This shows that n_m has increased by exactly one which balances the increment of n_e in eq. (3), the equation therefore remains valid in this case also.

In the third case (3°) the number of nodes remains unchanged also and we propose to show that the number of independent meshes increases by exactly one which shall then balance the increment in the number of elements and so eq. (3) shall remain valid in this case too.

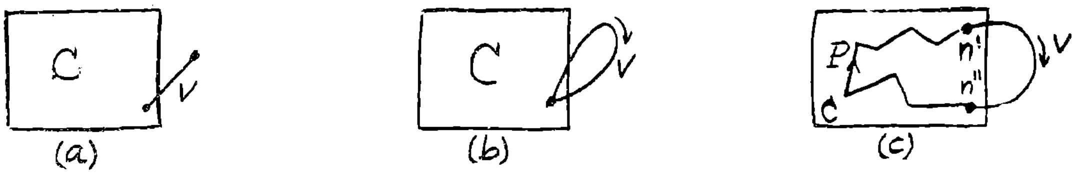


Fig. 3.

To do this assume the new element to be connected to the distinct nodes n' and n'' of C (Fig. 3c). Then since C is a connected part the nodes n' and n'' shall be mutually accessible and so there shall be a path through elements of C joining them, and hence there shall also (see Problem 4 in §3, p. 44) be a minimal (simple) path P joining them which together with the new element shall form <u>a</u> new mesh for which Kirchhoff's Voltage Law is $V_P + V = 0$, where V_P is the voltage drop from n'' to n' along P. Now consider <u>any</u> closed path in C'. If this closed path is wholly in C, the corresponding Kirchhoff's Voltage Law is already deducible from Kirchhoff's Voltage Laws for a complete independent set of meshes in C; and if the closed path contains the new element (so that the path is not wholly in C) the corresponding Voltage Law may be deduced from that of a closed path in C (obtainable by substituting the path P for the new element in the given closed path) by putting $V_P = -V$. Hence the Voltage Law for any closed path in C' follows from the Voltage Laws for a complete independent set of meshes in C together with the Voltage Law for the new mesh containing the new element, namely: $V+V_P=0$, while the latter is clearly independent from the meshes of C. This shows that the number of independent meshes in C' is exactly one more than the number of independent meshes in C. The increments of n_m and n_e thus balance out in eq. (3) which, therefore, remains valid in this case also and hence is valid in all cases.

If a network has n_c components, which may be numbered consecutively from 1 to n_c, we have, by eq. (3), for the component $a(=1,2,\dots,n_c)$, that:

$$n_{ma}+n_{na}-n_{ea} = 1, \qquad (a=1,2,\dots,n_c) \tag{4}$$

where n_{ma}, n_{na}, n_{ea} are the number of independent meshes, nodes, and elements, respectively, in the component <u>a</u>. Adding corresponding terms in the eqs. (4) for all the n_c components we get for the total number of independent meshes n_m, the total number of nodes n_n, and the total number of elements n_e, in the network, the following equations:

$$n_m+n_n-n_e=n_c \quad \text{or} \quad n_m=n_e-(n_n-n_c)=n_e-n_n' \quad \text{or} \quad n_m+n_n'=n_e \qquad (5)$$

where $n_n' = n_n-n_c$ is the total number of independent nodes in the network (= the total number of nodes less one node in each component of the network).

Problem 3. Considering that the removal of an essential element in a network decreases the number of independent meshes by exactly 1, show that the following function: $F = n_m+n_n-n_e-n_c$ remains constant with the removal of essential elements. After the removal of exactly n_m essential elements, then, the network shall be reduced to a network without meshes (the graph of each component of which is called a tree). In a tree it is easily shown that the number of nodes is equal to the number of elements plus 1, so that for a tree the number of independent nodes is equal to the number of elements (and so also for several trees). From this infer that $F=0$ and hence eqs. (5). (Cf. Prob. 4.)

Problem 4. For a collection of trees (i.e. a graph without meshes) show that the following function: $F'= n_n-n_c-n_e$ remains constant with the removal of the elements. By computing this constant when no element (or only one element with its terminals loose) is left, show that $F'=0$; thence infer that in the collection of trees the number of independent nodes and the number of elements are equal, i.e. that: $n_n-n_c=n_e$.

The proof given above of eqs. (3 & 4) for the detached parts (components) of a network (and of eq. (5) for the whole network) is constructive in the sense that it supplies effective means of constructing complete independent sets of meshes (for components of, and for, networks). The procedure is this: Choose an essential element and select a mesh containing it (in practice it is advisable to select as simple a mesh as possible). The chosen element will be called the <u>key element</u> of this mesh. Then imagine we remove the key element chosen from the network. In the reduced network repeat this demolishing <u>operation</u> of removing essential elements. After $n_m = n_e - n'_n$ steps we shall obtain a complete independent set of meshes of the original network, and what is left is naturally a network without meshes (whose graph is a collection of trees). The demolition <u>procedure</u> followed may be conveniently specified by the sequence of key elements chosen in the process, in the sense that this sequence of elements taken as key elements in succession shall defined (but not uniquely perhaps) a complete independent set of meshes (for each component of, and for, the whole network).

Once a complete independent set of meshes is chosen in a given oriented network (with n_e general elements and n'_n independent nodes) they can be (arbitrarily) oriented and numbered consecutively from 1 to $n_m = n_e - n'_n$. Kirchhoff's Voltage Law (eq. 1) may then be established for each of these meshes as follows:

$$\sum_{k=1}^{n_e} [k,m] V_k = 0, \qquad (m=1,2,\dots,n_m) \qquad (6)$$

and, according to our previous results, all of them are necessary and sufficient to ensure that Kirchhoff's Voltage Law shall be satisfied for each closed path through elements of the network.

To conclude this section on Kirchhoff's Voltage Law we add that when several elements are connected in parallel, with a graph as shown in Fig. 4, the whole combination may be considered as a unit (called a <u>parallel branch</u>) if all the voltage drops in the elements of the combination are denoted by a single symbol, as is justified by Kirchhoff's Voltage Law applied to the intermediate meshes of the combination.

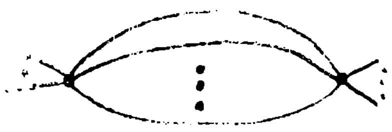


Fig. 4. The graph of elements connected in parallel.

CHAPTER III: THE SYSTEM OF EQUATIONS OF AN ARBITRARY NETWORK.

With the preparation of the preceding chapters we are now in a position to establish the complete system of equations necessary and sufficient to analyse any given (stationary) network. Once the system of equations is established the engineering problem is reduced to a mathematical problem to which the whole machinery of mathematics can be applied in order to study the performance of the network in regard to all the currents and voltages in its elements (with the help of which all such things as energies, powers, temperatures, and the like, may be studied). Since any network may be considered as a network of elements of the general series or parallel types it will be convenient to develop both aspects Then in any specific case the basic elements of a given network may be grouped into elements of the general series type, or into elements of the general parallel type, in order to study the network in the more convenient way.

§1. THE EQUATIONS OF A (STATIONARY) NETWORK OF GENERAL SERIES ELEMENTS.

Consider an arbitrary (stationary) network of n_e general series elements of the kind shown in Fig. 1. Let us arbitrarily assign a reference direction to each element and assume them to be arbitrarily numbered consecutively from 1 to n_e. Assume that the network has n_c components (separate parts) which may be numbered consecutively from 1 to n_c in any way whatsoever. If the network has a total of n_n nodes we can obtain a complete independent set of nodes by omitting exactly one node in each component; the rest may then be numbered consecutively from 1 to $n'_n = n_n - n_c$, in any way. Next let us choose a complete independent set of $n_m = n_e - n'_n = n_e - n_n + n_c$ meshes in the network, [redacted]he demolition process, as *explained in Ch. II, §4 (beginning at the bottom of p. 60), and let us arbitrarily assign reference directions (orientations) to (around) the meshes which we shall assume to be numbered consecutively from 1 to n_m in any way whatsoever. Then all the incidence numbers, (k,n) (for $k=1,2,\dots,n_e$ and $n=1,2,\dots,n_n')$ between the elements k and the nodes n, and $[k,m]$ (for $k=1,2,\dots,n_e$ and $m=1,2,\dots,n_m$) between the elements k and the meshes m, can be obtained from the network, by inspection, for the complete independent sets of nodes and meshes.*

Now let V_k denote the voltage drop and I_k the current in the element $k(=1,2,\dots,n_e)$ in the reference direction assigned to it.

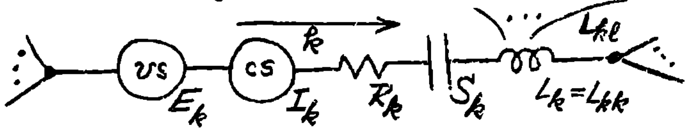


Fig. 1.

If E_k denotes the voltage rise in the voltage source and D_k denotes the voltage rise in the current source in the element k (in the assigned reference direction) we shall have (see Ch. II, §1, eq. 1):

$$V_k = V_{R_k} + V_{S_k} + V_{L_k} - E_k - D_k, \qquad (k=1,2,\dots,n_e) \qquad (1)$$

where V_{R_k}, V_{S_k}, V_{L_k}, denote the voltage drops in the resistor, condenser, and coil, respectively, in the element k. Denoting the resistance in the element k by R_k, the elastance by S_k, and the inductance by L_{kk} (or L_k), and the mutual inductance of the (coil in the) element k with (the coil in) the typical element l by L_{kl}, we shall have (by Ch. I, §§1-3):

$$V_{R_k} = R_k I_k, \qquad V_{S_k} = S_k\int I_k dt, \qquad V_{L_k} = \sum_{l=1}^{n_e} L_{kl}\frac{dI_l}{dt}. \qquad (k=1,2,\dots,n_e) \quad (2)$$

Hence, substituting in eq. (1), we get the following voltage equations for the voltage drops in the elements of the network:

$$V_k = R_k I_k + S_k\int I_k dt + \sum_{l=1}^{n_e} L_{kl}\frac{dI_l}{dt} - E_k - D_k. \qquad (3)$$
$$(k=1,2,\dots,n_e)$$

Besides these equations we have the equations expressing Kirchhoff's Current Law for the complete independent set of nodes (see Ch. II, §3, eq. 4, on p. 47):

$$\sum_{k=1}^{n_e}(k,n)I_k = 0; \qquad (n=1,2,\ldots,n_n') \qquad (4)$$

and the equations expressing Kirchhoff's Voltage Law for the complete independent set of meshes (see Ch. II, §4, eq. 6, p. 61):

$$\sum_{k=1}^{n_e}[k,m]V_k = 0. \qquad (m=1,2,\ldots,n_m) \qquad (5)$$

In all, the equations (3, 4 & 5) form a system of $n_e+n_n'+n_m$ equations; but by eq. (5) of Ch. II, §4, p. 60, we have that $n_n'+n_m=n_e$; hence the system has $2n_e$ equations. Moreover, the system also has $2n_e$ unknowns; because, if the generic element k has a current source then the corresponding voltage rise D_k in it and the voltage drop V_k between the terminals of the element are unknown (while the current I_k in it, being the value of a current source, is assumed to be known); and if the generic element k does not have a current source then the corresponding D_k is absent (or if one prefers, $D_k=0$ and is thus known), but then the current I_k and the voltage drop V_k in the element k are unknown; that is, in each element there are two unknown and so in the whole network (with n_e elements) there are $2n_e$ unknowns. [Since eqs.(4) establish n_n' linear relations between the n_e currents I_k, we may assume the number n_{cs} (say) of current sources in the network $\leq n_e-n_n'=n_m$; since otherwise some of the current sources would be superfluous, or inconsistent.] Considering the parameters of the network as indeterminates, then, the system of equations (3, 4 & 5) form a complete & independent system of $2n_e$ equations with $2n_e$ unknowns of the arbitrary (stationary) network with n_e general series elements. This system may then be considered as the general equations of any (stationary) network.

Note. When several voltage sources of a network are connected in parallel (so that their voltages must be equal in order to have consistency and the combination of which is also a voltage source) we shall always assume that they are replaced by a single (equivalent) voltage source of the same value. Also, when all the elements inciding on a certain node of a given network have current sources, we shall always assume them to have values consistent with Kirchhoff's Current Law for that node. We shall also assume that current sources of the same values in series are always replaced by a single (equivalent) current source of the same value, and that current sources in parallel are replaced by a single (equivalent) current source of a value equal to the (algebraic) sum of their values. Similarly, we shall also assume voltage sources in series to be replaced by a single (equivalent) voltage source of a value equal to the (algebraic) sum of their values. When an element consists only of a voltage source $\underline{E}$, the voltage drop $\underline{V}$ in this element is $-E$ and we say that the voltage drop $\underline{V}$ is an immediately soluble unknown, and similarly when $V=0$; and if all the elements of a certain mesh of a network consist only of voltage sources we shall always assume their values to be such as to satisfy Kirchhoff's Voltage Law for that mesh, in order to be consistent.

We will now work out an example to show how the complete system of equations of a rather complicated network can be established without difficulties, if we follow the indications given above.

Example. Consider the network shown in Fig. 2. First of all,

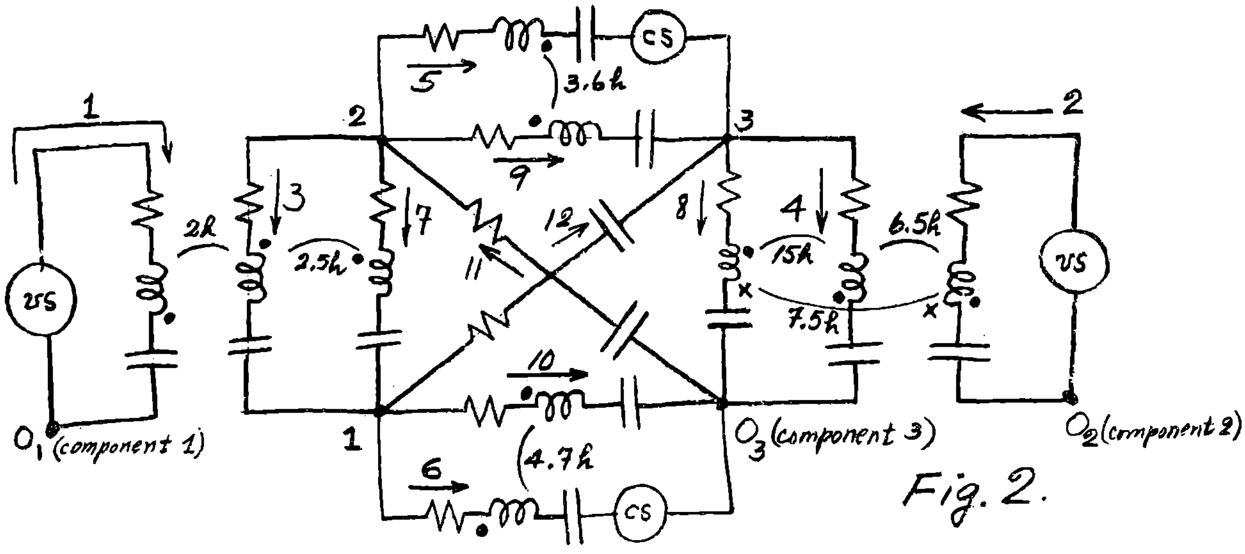


Fig. 2.

the basic elements are grouped into elements of the general series type. Of course, this may be done in many ways, and, in general, the number of elements of the general series type that result shall depend on the way the grouping is done. In Fig. 2, the basic elements were grouped into twelve such elements, demarked in the figure by heavy dots which are then taken as the nodes of the network; the elements were then arbitrarily oriented and numbered consecutively from 1 to 12 as shown. In the whole network there are six nodes and three components so that there are only 6-3=3 independent nodes. One base node was chosen in each component and these were marked O_1, O_2, O_3 in the figure; the rest of the nodes, which form a complete independent set of nodes in the network, were numbered 1, 2, 3, as shown. The number of independent meshes in the network is given by: $n_m=n_e-n'_n=12-3=9$. To choose nine independent meshes we first choose the key element 1, and as the first mesh the mesh formed by the element 1 alone, which we take as oriented in the sense of the element 1; next we choose the element 2 as key element, and as the second mesh the mesh formed by the element 2 alone, oriented in the sense of the element 2. In the third component we now take the element 3 as the key element for another mesh which we choose as the mesh $3,\bar{7}$, oriented in the sense of the key element 3 (the minus sign over the 7 indicating that the mesh traverses the element 7 in a sense opposite to that of its reference direction). Imagining all the key elements chosen up to now removed, we choose the element 4 as a key element for another mesh which we take as the mesh $4,\bar{8}$, oriented in the sense of the key element 4. Next we choose the mesh $5,\bar{9}$, with the key element 5; then the mesh $6,\overline{10}$, with the key element 6; then the mesh 7,10,11, with the key element 7. Next, with all the key elements chosen up to now imagined removed (and therefore only with the elements 8,9,10,11,12 left) we choose the element 9 as a key element for another mesh which we take as the mesh 9,8,11, oriented in the sense defined by this sequence (which in this case coincides with the sense of the key element 9). Finally, after further removing the element 9, the network remains with a single mesh which we take as the mesh $\bar{10}$,12,8, oriented in the sense defined by this sequence (which in this case is in the opposite sense as would be defined by the sense of the element 10 taken as the key element of the mesh; a fact, furthermore, indicated by the minus sign over the 10 in the sequence defining the mesh). We can be sure that the nine meshes chosen in this way, namely,

mesh 1: 1;
mesh 2: 2;
mesh 3: 3,$\bar{7}$;
mesh 4: 4,$\bar{8}$;
mesh 5: 5,$\bar{9}$;
mesh 6: 6,$\bar{10}$;
mesh 7: 7,10,11;
mesh 8: 9,8,11;
mesh 9: $\bar{10}$,12,8;

form a complete & independent set of meshes in the network of Fig. 2.

The voltage equations expressing the voltage drops in the elements of the network, given in general by eq. (3), are the following, in this specific network (the only non-zero mutual inductances being indicated by arcs between the corresponding coils):

$$V_1 = R_1 I_1 + S_1 \int I_1 dt + L_1 \frac{dI_1}{dt} + L_{13}\frac{dI_3}{dt} - E_1,$$

$$V_2 = R_2 I_2 + S_2 \int I_2 dt + L_2 \frac{dI_2}{dt} + L_{24}\frac{dI_4}{dt} + L_{28}\frac{dI_8}{dt} - E_2,$$

$$V_3 = R_3 I_3 + S_3 \int I_3 dt + L_3 \frac{dI_3}{dt} + L_{31}\frac{dI_1}{dt} + L_{37}\frac{dI_7}{dt},$$

$$V_4 = R_4 I_4 + S_4 \int I_4 dt + L_4 \frac{dI_4}{dt} + L_{42}\frac{dI_2}{dt} + L_{48}\frac{dI_8}{dt},$$

$$V_5 = R_5 I_5 + S_5 \int I_5 dt + L_5 \frac{dI_5}{dt} + L_{59}\frac{dI_9}{dt} - D_5,$$

$$V_6 = R_6 I_6 + S_6 \int I_6 dt + L_6 \frac{dI_6}{dt} + L_{6,10}\frac{dI_{10}}{dt} - D_6,$$

$$V_7 = R_7 I_7 + S_7 \int I_7 dt + L_7 \frac{dI_7}{dt} + L_{73}\frac{dI_3}{dt},$$

$$V_8 = R_8 I_8 + S_8 \int I_8 dt + L_8 \frac{dI_8}{dt} + L_{84}\frac{dI_4}{dt} + L_{82}\frac{dI_2}{dt},$$

$$V_9 = R_9 I_9 + S_9 \int I_9 dt + L_9 \frac{dI_9}{dt} + L_{95}\frac{dI_5}{dt},$$

$$V_{10} = R_{10} I_{10} + S_{10} \int I_{10} dt + L_{10} \frac{dI_{10}}{dt} + L_{10,6}\frac{dI_6}{dt},$$

$$V_{11} = R_{11} I_{11} + S_{11} \int I_{11} dt,$$

$$V_{12} = R_{12} I_{12} + S_{12} \int I_{12} dt,$$

where the L_{kl} are the (self and mutual) inductances referred to the reference directions shown in the figure 2.

Kirchhoff's Current Law equations are the following:

node 1: $-I_3+I_6-I_7+I_{10}+I_{12} = 0$,

node 2: $I_3+I_5+I_7+I_9-I_{11} = 0$,

node 3: $I_4-I_5+I_8-I_9-I_{12} = 0$.

Kirchhoff's Voltage Law equations are the following:

mesh 1: $V_1 = 0$, mesh 4: $V_4-V_8 = 0$, mesh 7: $V_7+V_{10}+V_{11}=0$,

mesh 2: $V_2 = 0$, mesh 5: $V_5-V_9 = 0$, mesh 8: $V_9+V_8+V_{11}=0$,

mesh 3: $V_3-V_7 = 0$, mesh 6: $V_6-V_{10}= 0$, mesh 9: $-V_{10}+V_{12}+V_8=0$.

The above equations, namely, the 12 voltage equations expressing the voltage drops in the 12 elements of the network, the 3 equations expressing Kirchhoff's Current Law for the set of independent nodes, and the 9 equations expressing Kirchhoff's Voltage Law for the set of independent meshes, constitute a complete & independent set of 24 equations with 24 unknowns (the 12 voltage drops $V_1,\ldots,V_{12}$, the voltage rises D_5 and D_6, and all the currents $I_1,\ldots,I_{12}$ except I_5 and I_6 which are the values of current sources and therefore known.

To say a few words about the signs of the mutual inductances in this example, let us suppose that the coils have the mutual inductances and the polarity marks shown in the figure (2, p. 65). Then, according to the rule of signs for the mutual inductances, given in Ch. I, §9, p. 28, we have: $L_{12}=-2h$, $L_{37}=+2.5h$, $L_{59}=-3.6h$, $L_{6,10}=+4.7h$, $L_{24}=+6.5h$, $L_{48}=-15h$, and $L_{28}=+7.5h$ (for which the polarity marks are the ✗). Had the reference direction assigned to the element 4 (for example) been directed from the node O_3 towards the node 3, instead of as shown in the figure, then L_{42} would be $-6.5h$ and L_{48} would be $+15h$; and the voltage equation for the voltage drop V_4' (say) in the new reference direction would be (where I_4' is the current in the new reference direction and L_{42}' & L_{48}' are, the new mutual inductances):

$$V_4'=R_4I_4'+S_4\int I_4'dt+L_4\frac{dI_4}{dt}+L_{42}'\frac{dI_2}{dt}+L_{48}'\frac{dI_8}{dt};$$

but since $V'_4=-V_4$, $I'_4=-I_4$, $L'_{42}=-L_{42}$, and $L'_{48}=-L_{48}$, while all the other quantities in this equation are the same as in the voltage equation for V_4, we infer that there has only been a change of sign in both members of the equation (which, therefore, remains essentially the same). On the other hand, the terms $L_{24}dI_4/dt$ and $L_{84}dI_4/dt$ in the voltage equations for V_2 and V_8, respectively, become $L'_{24}dI'_4/dt$ and $L'_{84}dI'_4/dt$ which are equal to the former; consequently the voltage equations for V_2 and V_8 remain unchanged. Thus all the voltage equations remain unchanged when the reference direction of the element 4 is changed (and the same can easily be shown for Kirchhoff's Current and Voltage Law equations); and this may be done with any element of the network. The same reasoning given here can obviously be used for any network, so that the network equations do not depend on the reference directions assigned to the elements. This is the reason why the reference directions can be assigned arbitrarily.

Concerning the graph of the network of this example we would like to say a few words too. The graph in the strict sense of the word is shown in Fig. 3a; the graph, considered as a collection of line segments with the proper identification of the terminals, is shown in Fig. 3b. The graph can also be exhibited in a more formal

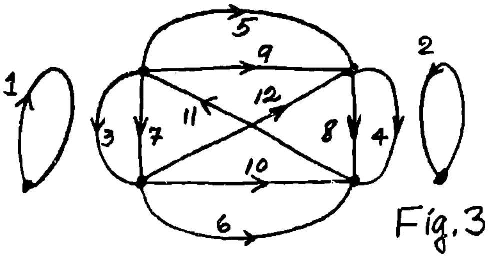


Fig. 3a.

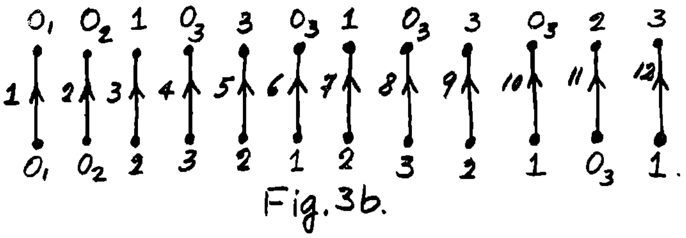


Fig. 3b.

way by giving the incidence numbers of the elements with the nodes. These may be suitably arranged in the form of a table, or $n_n \cdot n_e$ matrix:

$$
\begin{array}{r|cccccccccccc}
 & k= 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12\ (\text{elements}) \\
\hline
0_1 & \pm 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0_2 & 0 & \pm 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n=\quad 0_3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \\
(\text{nodes})\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 1 \\
2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 0 & -1
\end{array}
$$

Returning to the general equations (3, 4 & 5) of an arbitrary network we would like to mention that the number of unknowns may be readily reduced to half by eliminating the n_e voltage drops V_k upon substituting their values as given by eq. (3) into the equations (5) expressing Kirchhoff's Voltage Laws for a complete & independent set of meshes of the network. Another reduction in the number of unknowns may further be obtained by solving eq. (4), expressing Kirchhoff's Current Laws for a complete & independent set of nodes of the network, for n'_n unknown currents and then eliminating them by substitution in the previously obtained equations. In this way the number of unknowns is immediately reduced to n_m (the number of independent meshes of the network). ¶There is another way of accomplishing this result known as the <u>mesh method</u> which consists in making the following substitution (or change of variables):

$$I_k = \sum_{m=1}^{n_m} [k,m] J_m, \qquad (k=1,2,\ldots,n_e) \qquad (6)$$

in terms of new variables J_m $(m=1,2,\ldots,n_m)$, one for each mesh of a complete & independent set, known as <u>circulating</u> (or <u>mesh</u>) <u>currents</u>, first introduced by the great James Clerk Maxwell. The eqs. (4) expressing Kirchhoff's Current Law for the nodes of the network are then automatically satisfied on account of the theorem expressed by eq. (1) of Ch. II, §2 (p. 41), and thus drop out of the picture; but we shall not go in to this method any further at this point because it will be the object of another chapter.

<u>Problem 1</u>. Carry out the elimination process mentioned above, in the case of the example given previously (concerning Fig. 2).

<u>Problem 2</u>. An electric machine may be considered as a network of <u>N</u> elements of the kind shown in the Fig. 4, some of which move as a rigid body around an axis which is fixed with respect to the rest.

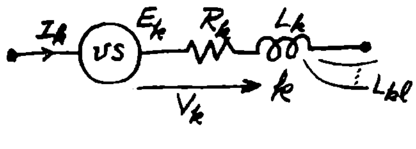


Fig. 4.

With the notation of Prob. 3, Ch. I, §3, (p.8) show that the general equations of an electric machine are the following:

$$V_k = R_k I_k + \sum_{l=1}^{N} L_{kl}\frac{dI_l}{dt} + \omega\frac{\partial\psi_k}{\partial\vartheta} + \frac{\partial\psi_k}{\partial t} - E_k, \quad (k=1,2,\cdots,N)$$

$$\sum_{k=1}^{N}(k,n)I_k = 0, \quad (n=1,2,\cdots,n_n'), \quad \sum_{k=1}^{N}[k,m]V_k = 0, \quad (m=1,2,\cdots,n_m), \quad T = M\frac{d\omega}{dt},$$

where T is the torque on the rotor and M is the moment of inertia of the rotor about the axis of rotation. In this system of 2N+1 equations with 2N+1 unknowns (the N voltages V_k, the N currents I_k, and the angle of rotation ϑ) the inductances L_{kl} and the incidence numbers (k,n) and $[k,m]$ vary, in general, with the angle ϑ.

§2. THE EQUATIONS OF A (STATIONARY) NETWORK OF PARALLEL ELEMENTS.

Consider an arbitrary (stationary) network of n_e general parallel elements of the kind shown in Fig. 1. Let us arbitrarily assign a reference direction to each element and number them consecutively from 1 to n_e in any way whatsoever. Assume that the network has n_n nodes and n_c components (separate parts); then omitting exactly one node in each component (which may be marked 0_1, 0_2, ..., 0_{n_c}) let us number the rest consecutively from 1 to $n_n' = n_n - n_c$, arbitrarily. Next, let us choose any complete & independent set of $n_m = n_e - n_n'$ meshes in the network (as explained in Ch. II, §4), assign an arbitrary reference direction to (around) each, and number them consecutively from 1 to n_m, in any way whatever. All the incidence numbers (k,n) (for $k=1,2,\ldots,n_e$ and $n=1,2,\ldots,n_n'$) between the elements k and the nodes n, and $[k,m]$ (for $k=1,2,\ldots,n_e$ and $m=1,2,\ldots,n_m$) between the elements k and the meshes m, can then be obtained by inspection from the network for all the nodes and meshes of the complete and independent sets of nodes and meshes chosen, respectively.

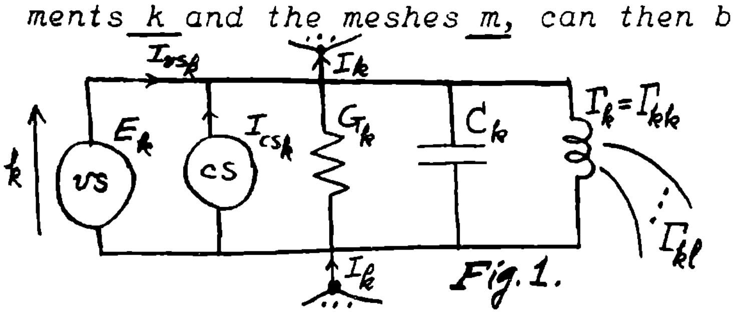


Fig. 1.

Now let G_k denote the conductance, C_k the capacitance, and $\Gamma_k = \Gamma_{kk}$ the (self) invertance (inverse inductance, in the presence of all the other coils) in the element k; and let Γ_{kl} denote the mutual invertance (mutual inverse inductance) of the (coil in the) element k with the (coil in the) generic element l (in the presence of all the other coils). Also, let V_k denote the voltage drop and I_k the total (terminal) current through the element $k(=1,2,\dots,n_e)$ in the reference direction assigned to it; let I_{cs_k} denote the value of (i.e. the prescribed current in) the current source, and let I_{vs_k} denote the current in the voltage source (whose value, i.e., its prescribed voltage rise, will be denoted by E_k), in the assigned reference direction. Then we shall have (cf. Ch. II, §1, eq. 2):

$$I_k = I_{vs_k} + I_{cs_k} + I_{G_k} + I_{C_k} + I_{\Gamma_k}, \quad (k=1,2,\dots,n_e) \quad (1)$$

where I_{G_k}, I_{C_k}, and I_{Γ_k}, denote the currents in the resistor, condenser, and coil, respectively. Now, by Ch. I, §§1-3, we have:

$$I_{G_k} = G_k V_k, \quad I_{C_k} = \frac{d}{dt}(C_k V_k), \quad I_{\Gamma_k} = \sum_{l=1}^{n_e} \int \Gamma_{kl} V_l \, dt. \quad (k=1,2,\dots,n_e)(2)$$

Hence, substituting in eq. (1), we get the following current equations for the total (terminal) currents through the elements:

$$I_k = G_k V_k + \frac{d}{dt}(C_k V_k) + \sum_{l=1}^{n_e} \int \Gamma_{kl} V_l \, dt + I_{cs_k} + I_{vs_k}, \quad (k=1,2,\dots,n_e)(3)$$

or, if all the C_k and all the Γ_{kl} are constant:

$$I_k = G_k V_k + C_k \frac{d}{dt} V_k + \sum_{l=1}^{n_e} \Gamma_{kl} \int V_l \, dt + I_{cs_k} + I_{vs_k}. \quad (k=1,2,\dots,n_e) \quad (4)$$

Besides these equations we have the equations expressing Kirchhoff's Current Law for the complete independent set of nodes (see Ch. II, §3, eq. 4, on p. 47):

$$\sum_{k=1}^{n_e} (k,n) I_k = 0; \quad (n=1,2,\dots,n_n') \quad (5)$$

and the equations expressing Kirchhoff's Voltage Law for the complete independent set of meshes (see Ch. II, §4, eq. 6, on p. 61):

$$\sum_{k=1}^{n_e} [k,m] V_k = 0. \quad (m=1,2,\dots,n_m) \quad (6)$$

The equations (3, 5, 6) form a system of $n_e+n'_n+n_m = 2n_e$ *equations with* $2n_e$ *unknowns; if the generic element* _k_ *has a voltage source then the corresponding current through it,* I_{vs_k}, *and the total (terminal) current through the element,* I_k, *are unknown (while the voltage drop* V_k *in the element* _k_ *is equal to the negative value of the voltage source,* $-E_k$, *and is therefore assumed to be known), and if the generic element* _k_ *does not have a voltage source then the corresponding* I_{vs_k} *is absent (or if one prefers,* $I_{vs_k}=0$ *and is thus known) but then the voltage drop* V_k *and the total (terminal) current* I_k *through the element* _k_ *are unknown; thus, in any case, there are two unknowns per element so that in all the* n_e *elements of the network there are* $2n_e$ *unknowns.* [Since eqs.(6) establish n_m linear relations between the n_e voltages V_k, we may assume the number n_{vs}(say) of voltage (E) sources in the network (=number of known V_k) $\leq n_e-n_m=n'_n$, since otherwise some of the voltage sources would be superfluous, or inconsistent.] *(If the element consists only of a current source then* $I_k=I_{cs_k}$ *and we say that* I_k *is an immediately soluble unknown.* And similarly when $I_k=0$.*) Considering the parameters of the network as indeterminates, then, the system of equations (3, 5 & 6) form a complete & independent system of* $2n_e$ *equations with* $2n_e$ *unknowns of the arbitrary (stationary) network with* n_e *general parallel elements. This system can then be considered as the* _general equations of any (stationary) network_ *also.*

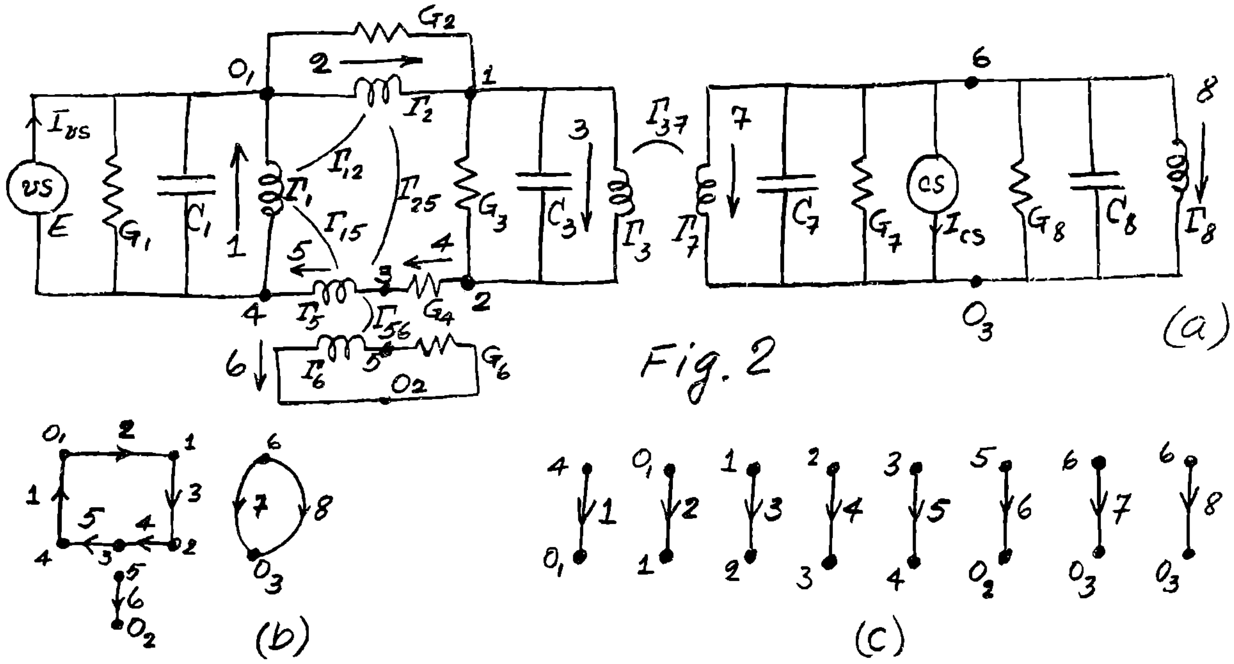


Fig. 2

(b) (c)

Example. Consider the network shown in Fig. 2a, the graph of which is shown in (b), in the usual sense, and in (c), as a collection of segments with the proper identification of the terminals, after the basic elements were grouped into eight elements of the general parallel type as demarked in the figure by heavy dots (the nodes of the network). There are nine nodes and three components (separate parts), so that there are 9-3=6 independent nodes, and 8-6=2 independent meshes. The set of independent nodes chosen has the nodes marked 1,2,3,4,5,6; the omitted nodes (i.e. the base nodes) are the ones marked $0_1, 0_2, 0_3$. The only meshes of the network are: the mesh 12345 and the mesh $7\bar{8}$, as can be appreciated better from the graph shown in (b), of Fig. 2.

The eight current equations expressing the total (terminal) currents in the elements of the network, I_k (k=1,2,...,8), are:

$$I_1 = I_{vs} + G_1 V_1 + \frac{d}{dt}(C_1 V_1) + \int \Gamma_1 V_1 dt + \int \Gamma_{12} V_2 dt + \int \Gamma_{15} V_5 dt,$$
$$I_2 = G_2 V_2 + \int \Gamma_2 V_2 dt + \int \Gamma_{21} V_1 dt + \int \Gamma_{25} V_5 dt,$$
$$I_3 = G_3 V_3 + \frac{d}{dt}(C_3 V_3) + \int \Gamma_3 V_3 dt + \int \Gamma_{37} V_7 dt,$$
$$I_4 = G_4 V_4,$$
$$I_5 = \int \Gamma_5 V_5 dt + \int \Gamma_{51} V_1 dt + \int \Gamma_{52} V_2 dt + \int \Gamma_{56} V_6 dt,$$
$$I_6 = G_6 V_6 + \int \Gamma_6 V_6 dt + \int \Gamma_{65} V_5 dt,$$
$$I_7 = I_{cs} + G_7 V_7 + \frac{d}{dt}(C_7 V_7) + \int \Gamma_7 V_7 dt + \int \Gamma_{73} V_3 dt,$$
$$I_8 = G_8 V_8 + \frac{d}{dt}(C_8 V_8) + \int \Gamma_8 V_8 dt,$$

where the $\Gamma_{k\ell}$ are the (self and mutual) invertances (inverse inductances) referred to the reference directions shown in the figure 2. Notice that the serie branch formed by the elements 4 & 5 cannot be considered as a single element of the parallel type; notice also that the basic elements of the component 3 (the one with the base node O_3) have to be grouped into two elements of the parallel type (which of course can be done in several ways) unless the whole parallel branch is first reduced to a single parallel element.

The equations expressing Kirchhoff's Current Law for the nodes 1 to 6 of the complete and independent set of nodes are:

node 1: $-I_2+I_3 = 0$, node 4: $I_1-I_5 = 0$,
node 2: $-I_3+I_4 = 0$, node 5: $I_6 = 0$,
node 3: $-I_4+I_5 = 0$, node 6: $I_7+I_8 = 0$,

as can be appreciated better from the graph shown in Fig. 2(b).

The equations expressing Kirchhoff's Voltage Law for the two meshes of the complete & independent set of meshes are:

mesh 1: $V_1+V_2+V_3+V_4+V_5 = 0$, mesh 2: $V_7-V_8 = 0$.

The above equations, namely, the 8 current equations expressing the total (terminal) currents in the 8 elements of the network, the 6 equations expressing Kirchhoff's Current Law for the set of independent nodes, and the 2 equations expressing Kirchhoff's Voltage Law for the set of independent meshes, form a complete & independent set of 16 equations with 16 unknowns which determine the behavior of the given network completely. (The 16 unknowns are: The 8 terminal currents $I_1, I_2, \ldots, I_8$, although I_6 is immediately soluble from Kirchhoff's Current Law applied to the node 5, the 7 voltage drops $V_2, V_3, \ldots, V_8$, and the current I_{vs} through the vs.)

This network (shown in Fig. 2) can of course be treated as a network of elements of the general series type by means of the general equations (3, 4 & 5) of §1; but the treatment would be much more lengthly, because ~~it~~ the network would have to be considered as having 18 elements of the serie type connected into three components with seven nodes (with 7-3=4 independent nodes and 18-4=14 independent meshes) and so the system of equations would have 36 equations with 36 unknowns (which could be reduced however immediately to 14).

Returning to the general equations (3, 5 & 6) of an arbitrary network, the number of unknowns may readily be reduced to half by eliminating the n_e terminal currents I_k upon substituting their values as given by eq. (3) into the eq. (5) expressing Kirchhoff's Current Law for a complete & independent set of nodes of the network. Another reduction in the number of unknowns may further be obtained by solving eq. (6), expressing Kirchhoff's Voltage Law for a complete & independent set of meshes of the network, for n_m unknown voltage drops and then eliminating them by substitution in the previously obtained equations. In this way the number of unknowns is immediately reduced to n'_n (the number of independent nodes of the network). ¶There is another way of accomplishing this result known as the <u>*node*</u> <u>*method*</u> *which consists in making the following substitution (or change of variables):*

$$V_k = \sum_{n=1}^{n'_n} (k,n)U_n, \qquad (k=1,2,\ldots,n_e) \qquad (7)$$

in terms of the potentials U_n (introduced in Ch. II, §4, p. 53) as new variables, one for each node of a complete & independent set of nodes of the network (see Problem 1 of Ch. II, §4, p. 53). Then the eq. (6), expressing Kirchhoff's Voltage Law for the meshes of the network, are automatically satisfied on account of the theorem expressed by eq. (1) of Ch. II, §2 (p. 41), and thus drop out of the picture. We shall not go into this method any further at this point because it will be the object of another chapter.

<u>*Problem 1*</u>. *Carry out the elimination process, mentioned above, for the network of Fig. 2 considered in the example given previously.*

<u>*Problem 2*</u>. *Establish the equations for the networks whose graphs are shown in Fig. 3, assuming that all the elements are of the general parallel type and that only the elements 1 and 6 have sources and that these sources are current sources.*

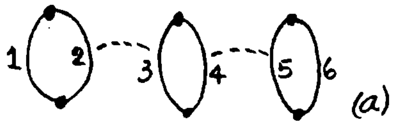
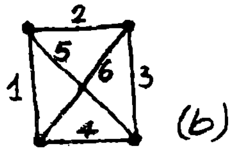


Fig. 3. In (a) the non-zero mutual inductances are indicate by dotted lines; in (b) there are no coils.



§3. SPECIAL CASES.

a). Networks without mutual inductances. When a network has no mutual inductances, $L_{kl}=0$ and $\Gamma_{kl}=0$ for all $k \neq l$. Therefore the general equations of the network are:

$$V_k = R_kI_k+L_kdI_k/dt +S_k\int I_kdt \quad E_k - D_k, \quad (k=1,2,\ldots,n_e),$$

$$\sum_{k=1}^{n_e}(k,n)I_k = 0, \quad (n=1,2,\ldots,n_n'), \quad \sum_{k=1}^{n_e}[k,m]V_k = 0, \quad (m=1,2,\ldots,n_m), \qquad (1)$$

if the network is considered as a network of elements of the general series type (see eq. 3, 4, 5 of §1, pp. 63-4); and:

$$I_k = G_kV_k+d(C_kV_k)/dt +\int\Gamma_kV_kdt +I_{cs_k}+I_{vs_k}, \quad (k=1,2,\ldots,n_e)$$

$$\sum_{k=1}^{n_e}(k,n)I_k=0, \quad (n=1,2,\ldots,n_n'), \quad \sum_{k=1}^{n_e}[k,m]V_k=0, \quad (m=1,2,\ldots,n_m), \qquad (2)$$

if the network is considered as a network of elements of the general parallel type (see eq. 3, 5, 6 of §2, p. 72).

b). Direct current (d-c) networks. When all the (current and voltage) sources of a network have constant values (i.e. when all the exciting functions are constant) the network can be considered, in a rather generalized sense, as a d-c network; and the general equations of such a network can be obtained from the general equations of an arbitrary network simply by making all the exciting functions constant. However, the theory of d-c networks is usually restricted to networks all of whose parameters (resistances, inductances and capacitances) and exciting functions (the values of sources) are constant and for which one seeks only the constant (parts of the) response quantities, that is, the constant values of the unknown currents and voltages which satisfy the system of equations of the network (the constant solutions). Therefore, from this accepted point of view, the general equations of an arbitrary d-c network can be obtained from the general equations of an arbitrary network by making all the quantities in these equations constant. (For this reason these networks could be called, more properly, constant current (c-c) networks.) As a consequence all derivatives in these equations disappear (as if there were no inductances). Considering the d-c network as a network of series elements, the equations then reduce to the following:

$$V_k = R_k I_k + S_k \int I_k dt - E_k - D_k, \quad (k=1,2,\dots,n_e)$$

$$\sum_{k=1}^{n_e}(k,n)I_k=0, \quad (n=1,2,\dots,n'_n) \qquad \sum_{k=1}^{n_e}[k,m]V_k=0, \quad (m=1,2,\dots,n_m) \tag{3}$$

where all the R_k, S_k, E_k, D_k, V_k, and I_k are constant. Performing the integrations in the first of these equations we get:

$$V_k = R_k I_k + S_k I_k t + V_k^\circ - E_k - D_k, \quad (k=1,2,\dots,n_e)$$

where the V_k° are the integration constants $(k=1,2,\dots,n_e)$. Equalizing corresponding coefficients (of the time t) in this equation we get:

$$V_k = R_k I_k + V_k^\circ - E_k - D_k \quad \& \quad S_k I_k = 0; \; (k=1,2,\dots,n_e)$$

hence if the element <u>k</u> has a condenser (meaning that $S_k \neq 0$) we infer that $I_k=0$, so that the branch of a d-c network with a condenser will behave like an open circuit, and so can have no current source in it with a value $\neq 0$; consequently, omitting the null current sources, the expression for the voltage drop in an element <u>k</u> with a condenser reduces to the rather trivial expression which follows (V_k° being the constant voltage drop across the condenser):

$$V_k = V_k^\circ - E_k \quad \text{(for the elements } \underline{k} \text{ with condensers);} \tag{4}$$

on the other hand, if the element <u>k</u> does not have a condenser, the term V_k° is absent and the equation for V_k becomes:

$$V_k = R_k I_k - E_k - D_k \quad \text{(for the elements } \underline{k} \text{ without condensers).} \tag{5}$$

Summarizing, in a d-c network all the coils may be ignored and considered as short circuits while all the condensers may also be ignored and considered as open circuits; the d-c network may then be considered as a network of resistances and constant (voltage and current) sources alone, from the accepted point of view.

c). <u>Alternating</u> <u>current</u> <u>(a-c)</u> <u>networks</u>. In a rather generalized sense, the theory of alternating currents is concerned with networks with constant parameters in which all the exciting functions (i.e. the values of the voltage and current sources) are of the following form:

$$\sum_n \left[p_n(t)\sin\omega_n t + q_n(t)\cos\omega_n t\right], \qquad (6)$$

where all the $p_n(t)$ and $q_n(t)$ are polynomials in the time t. It can then be shown that all the response quantities (the unknown currents and voltages) are also of this same form, with perhaps additional ω_n corresponding to what are known as the generalized (complex) natural (angular) frequencies of the network; however it is accepted that the theory of alternating currents is restricted to the case in which all the polynomials $p_n(t)$ and $q_n(t)$ are constant, and in which all the response quantities are (assumed to be) of the same form as the exciting functions (with no additional ω_n). In either case the theory may be developed by substituting functions of the proper form into the general equations of an arbitrary network and then making use of the linear independence (cf. Ch. IV, §7, p.111) of functions of the form $t^m e^{at}$ (where m is a non-negative integer and a is any complex number), or of the forms $t^m \sin(at)$ and $t^m \cos(at)$, when the m's or the a's are distinct, in order to split the general equations into various systems of equations corresponding to the various (pairs of) values of m and a. (The details of all this will be the object of following chapters.) A general problem that then makes its natural appearance is that of a (system of equations corresponding to a) network in which all the currents and voltages are sine functions of the same angular frequency. As a matter of fact this problem is of such a general character that almost all beginners think for a long time after that it alone is the object of alternating current theory; but this is natural, because by far the greater parts of (and all of some of the) books on alternating currents are devoted to this problem, which may be called, more properly, the problem of an arbitrary network in the sinusoidal state; and the theory may be referred to as that on sinusoidal current (s-c) networks. We shall make no exception, either, and most of the following chapters will also be devoted to the theory of networks in the sinusoidal state.

The *general equations of a network in the sinusoidal state* can be obtained from the general equations of an arbitrary network by substituting sine functions of the same frequency for all the (exciting and response) currents and voltages and considering all the parameters of the network to be constant. (At this point we will only give an anticipated outline of what will be treated in detail in another chapter.) The integration constants that make their appearance in these equations may be eliminated by differentiation and, then, all the terms that remain in the resulting equations will be sinusoids of the same frequency, of the form (say):

$$F(t) = A.\sin(\omega t+a) = A'\sin \omega t + A''\cos \omega t = \frac{A}{2i}(e^{ia}e^{i\omega t} - e^{-ia}e^{-i\omega t}),$$

(by the well-known formulae due to Euler) where $\omega>0$ and $A' = A.\cos \underline{a}$ and $A'' = A.\sin \underline{a}$. The angular frequency, ω, is the same for all the sinusoids and so each sinusoidal current and voltage shall be determined completely by the corresponding (ordered) pair of numbers A' and A'' (and also by the coefficient A and the angle $\underline{a}$), which in turn are completely determined by the complex number $A'+A''i$ (and also by the complex number Ae^{ia}) which will be denoted by F^*, as corresponding to the sinusoid $F(t)$. If we now imagine all the currents and voltages in the general equations of an arbitrary network to be replaced by sine functions of the same angular frequency ω, omit all integration constants, multiply by $e^{i\omega t}$ (or by $e^{-i\omega t}$), then differentiate again (with respect to the time $\underline{t}$), and finally make use of the fact that $e^{2i\omega t} \neq 0$ (or that $e^{-2i\omega t} \neq 0$) for all $\underline{t}$, we get the following equations (for a general s-c network of series elements):

$$V_k^* = R_kI_k^* + S_kI_k^*/i\omega + \sum_{l=1,\cdots,n_e} i\omega L_{kl}I_l^* - E_k^* - D_k^*, \quad (k=1,2,\ldots,n_e)$$
$$\sum_{k=1}^{n_e}(k,n)I_k^*=0, \quad (n=1,2,\ldots,n_n') \qquad \sum_{k=1}^{n_e}[k,m]V_k=0, \quad (k=1,2,\ldots,n_m) \tag{7}$$

from which all the unknown complex numbers V_k^*, D_k^*, and the unknown I_k^*, can be determined. Once these complex numbers are determined, all the corresponding sinusoids V_k, D_k, and I_k, can be reconstructed as sine functions of the time $\underline{t}$, all of the same angular frequency ω. Thus all the unknown sinusoidal currents and voltages can be determined from eq.(7), which may then be called the *general equations of an arbitrary network* (of series elements) *in the sinusoidal state*.

CHAPTER IV: COMPLEX NUMBERS AND SINE FUNCTIONS.

In this chapter we will develop the mathematical material necessary to undertake the proper treatment of an arbitrary network in the sinusoidal state, in the next chapter.

§1. COMPLEX NUMBERS.

Assume that we know how to deal with real numbers. Let x and y denote any real numbers and consider the totality of ordered pairs $\langle x,y\rangle$. Two ordered pairs, $\langle x',y'\rangle$ and $\langle x'',y''\rangle$, of real numbers will be considered equal if and only if $x'=x''$ and $y'=y''$. The negative, $-\langle x,y\rangle$, of the ordered pair $\langle x,y\rangle$ is defined to be the ordered pair $\langle -x,-y\rangle$. The sum of two ordered pairs: $\langle x',y'\rangle$ and $\langle x'',y''\rangle$, denoted by $\langle x',y'\rangle + \langle x'',y''\rangle$, is defined to be the ordered pair $\langle x'+x'',y'+y''\rangle$.

Problem 1. Show that:

$\langle x',y'\rangle + \langle x'',y''\rangle = \langle x'',y''\rangle + \langle x',y'\rangle$, (the commutative law of addition)

$\langle x,y\rangle + \langle 0,0\rangle = \langle x,y\rangle$, (the rôle of the null ordered pair)

$\langle x,y\rangle + [-\langle x,y\rangle] = \langle 0,0\rangle$ (the rôle of the negative ordered pair)

$\langle x,y\rangle + [\langle x',y'\rangle + \langle x'',y''\rangle] = [\langle x,y\rangle + \langle x',y'\rangle] + \langle x'',y''\rangle$,

the latter being referred to as the associative law of addition.

The difference, $\langle x',y'\rangle - \langle x'',y''\rangle$, of two ordered pairs is defined to be an ordered pair, $\langle x,y\rangle$ (say), such that $\langle x',y'\rangle = \langle x'',y''\rangle + \langle x,y\rangle$. If this difference exists it must be unique, because: $\langle x'',y''\rangle + \langle x,y\rangle = \langle x'',y''\rangle + \langle x^*,y^*\rangle$ implies that $x''+x=x''+x^*$ and $y''+y=y''+y^*$ and hence that $x=x^*$ and $y=y^*$, and consequently that $\langle x,y\rangle = \langle x^*,y^*\rangle$. Moreover, by direct substitution, it can be shown that the difference is equal to $\langle x',y'\rangle + [-\langle x'',y''\rangle] = \langle x'-x'',y'-y''\rangle$, as follows:

$$\langle x'',y''\rangle + \langle x'-x'',y'-y''\rangle = \langle x''+(x'-x''),y''+(y'-y'')\rangle = \langle x',y'\rangle .$$

Consequently the difference always exists and is unique.

Problem 2. Show that: $\langle x',y'\rangle - [-\langle x'',y''\rangle] = \langle x',y'\rangle + \langle x'',y''\rangle$.

The product of two ordered pairs: $\langle x',y'\rangle$ and $\langle x'',y''\rangle$, denoted by $\langle x',y'\rangle\langle x'',y''\rangle$, is defined to be the ordered pair $\langle x'x''-y'y'', x'y''+x''y'\rangle$. The product of $\langle x,y\rangle$ by itself is called its square, denoted $\langle x,y\rangle^2$.

Problem 3. Show that:

$\langle x',y'\rangle\langle x'',y''\rangle = \langle x'',y''\rangle\langle x',y'\rangle$, (commutative law of mutiplication)

$\langle x,y\rangle\langle 1,0\rangle = \langle x,y\rangle$, (the rôle of $\langle 1,0\rangle$ is that of a unit)

$\langle x,y\rangle\langle 0,0\rangle = \langle 0,0\rangle$, (the rôle of $\langle 0,0\rangle$ in multiplication)

$\langle 0,1\rangle\langle 0,1\rangle = \langle 0,1\rangle^2 = -\langle 1,0\rangle$, (the square of $\langle 0,1\rangle$ is the negative of the unit)

$\langle x,y\rangle\left[\langle x',y'\rangle\langle x'',y''\rangle\right] = \left[\langle x,y\rangle\langle x',y'\rangle\right]\langle x'',y''\rangle$,

the latter being referred to as the associative law of mutiplication.

Problem 4. Show that:

$$\left[-\langle x,y\rangle\right]\left[-\langle x',y'\rangle\right] = \langle x,y\rangle\langle x',y'\rangle ,$$

$$\langle x,y\rangle\left[-\langle x',y'\rangle\right] = \left[-\langle x,y\rangle\right]\langle x',y'\rangle = -\left[\langle x,y\rangle\langle x',y'\rangle\right] .$$

Problem 5. Show that if: $\langle x,y\rangle\langle x',y'\rangle = \langle 0,0\rangle$, then: $\langle x,y\rangle = \langle 0,0\rangle$ or $\langle x',y'\rangle = \langle 0,0\rangle$. From this deduce that $\langle 1,0\rangle$ is the only unit.

Problem 6. Show that we have the distributive law, namely:

$$\langle x,y\rangle\left[\langle x',y'\rangle + \langle x'',y''\rangle\right] = \langle x,y\rangle\langle x',y'\rangle + \langle x,y\rangle\langle x'',y''\rangle .$$

The quotient, $\langle x',y'\rangle/\langle x'',y''\rangle$, of two ordered pairs is defined, only if $\langle x'',y''\rangle \neq \langle 0,0\rangle$, as the ordered pair $\langle x,y\rangle$ (say) such that: $\langle x',y'\rangle = \langle x'',y''\rangle\langle x,y\rangle$. When the quotient exists it must be unique; because if $\langle x^*,y^*\rangle$ is also the quotient we have: $\langle x',y'\rangle = \langle x'',y''\rangle\langle x,y\rangle$ and $\langle x',y'\rangle = \langle x'',y''\rangle\langle x^*,y^*\rangle$ and so: $\langle x'',y''\rangle\langle x,y\rangle = \langle x'',y''\rangle\langle x^*,y^*\rangle$, and hence: $\langle x'',y''\rangle\left[\langle x,y\rangle - \langle x^*,y^*\rangle\right] = \langle 0,0\rangle$; but $\langle x'',y''\rangle \neq 0,0$, by hypothesis, consequently: $\langle x,y\rangle - \langle x^*,y^*\rangle = \langle 0,0\rangle$; that is: $\langle x,y\rangle = \langle x^*,y^*\rangle$. Moreover, by direct substitution it can easily be shown that: $\left\langle \frac{x''}{x''^2+y''^2}, \frac{-y''}{x''^2+y''^2}\right\rangle\langle x',y'\rangle$, is the quotient $\langle x',y'\rangle/\langle x'',y''\rangle$. Thus the quotient of two ordered pairs always exists and is unique, if the denominator is not the null ordered pair.

Problem 7. The inverse of an ordered pair: $\langle x,y\rangle \neq \langle 0,0\rangle$, denoted $\langle x,y\rangle^{-1}$, is defined to be the quotient: $\langle 1,0\rangle/\langle x,y\rangle$. Show that:

$$\langle x,y\rangle^{-1}\langle x,y\rangle = \langle x,y\rangle\langle x,y\rangle^{-1} = \langle 1,0\rangle \text{ (the unit), and: } \langle x,y\rangle^{-1} = \left\langle \frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right\rangle.$$

Problem 8. If $\langle x,y\rangle \neq \langle 0,0\rangle$ show that:

$$\langle x',y'\rangle / \langle x,y\rangle = \langle x',y'\rangle \langle x,y\rangle^{-1} = \langle x,y\rangle^{-1} \langle x',y'\rangle = \left\langle \frac{x'x+y'y}{x^2+y^2}, \frac{xy'-x'y}{x^2+y^2} \right\rangle.$$

We would now like to immerse the real numbers in the set of ordered pairs. For this, let us consider the following correspondence, between the set of real numbers x and the set of ordered pairs $\langle x,0\rangle$ with the second term zero: $\langle x,0\rangle \longleftrightarrow x$. This is obviously a one to one correspondence, and it is not difficult to show that it preserves addition, subtraction, multiplication, and division (if defined). By this we mean that:

$$\langle x,0\rangle \pm \langle x',0\rangle \longleftrightarrow x \pm x', \qquad \langle x,0\rangle \langle x',0\rangle \longleftrightarrow xx', \qquad \langle x,0\rangle / \langle x',0\rangle \longleftrightarrow x/x' \ (\text{if } x' \neq 0).$$

Technically, this correspondence is called an isomorphism between the real numbers x and the ordered pairs $\langle x,0\rangle$ with respect to the relations (or operations) which it preserves; and on the basis of this we now take the liberty of replacing all the ordered pairs of the form $\langle x,0\rangle$ by the corresponding real number x, (in all operations and relations. Thus in the new system we will) define all the operations between a real number x and an ordered pair $\langle x',y'\rangle$, with $y' \neq 0$, as the corresponding operation between $\langle x,0\rangle$ and $\langle x',y'\rangle$, if the result is not of the form $\langle x'',0\rangle$; and if it is of the form $\langle x'',0\rangle$, we define it as x'' (x'' = real); and, similarly, if an operation between two ordered pairs has a result of the form $\langle x'',0\rangle$ we define the operation to have the result x'' (= real).

This process of substituting the real numbers for the corresponding ordered pairs with null second terms in all operations and relations amongst ordered pairs, is called immersing or imbedding the real numbers in the system of ordered pairs. The resulting system (so constructed), of real numbers and ordered pairs with non-zero second terms, is called the complex number system. It is also called the (open) complex (number) plane, for (conventional) reasons to appear shortly. When an extra symbol, ∞, is included (whose relation to the complex numbers is irrelevant here), called the point *at infinity*, the augmented system is called the *closed complex* (number) *plane*, or the *complex* (number) *sphere*.

Problem 9. Show that in the complex number system we have:

$$a\langle x,y\rangle = \langle ax,ay\rangle, \text{ (where } a \text{ is any real number)}$$

$$\langle 0,1\rangle^2 = \langle 0,1\rangle\langle 0,1\rangle = -1.$$

The complex number: $\langle 0,1\rangle$ is conveniently denoted by i, so that we have: $i^2=-1$. (In many books on electrical engineering, j is used instead of i; because the letter i is used for currents (as if there were no more symbols that could be used); but we prefer to use i because it is universally accepted so in mathematics and, after all, complex numbers do belong primarily to mathematics.) Therefore, unlike with real numbers, the square of i is a negative number. For this reason, any complex number of the form ai (a=real) is (conventionally) called an *imaginary number*, and i itself is called the *imaginary unit*.

Since, for any ordered pair $\langle x,y\rangle$ of the complex number system, we have: $\langle x,y\rangle = x + \langle 0,y\rangle = x + y\langle 0,1\rangle = x+yi$, and since for any real number x we have: $x=x+0i$, we see that the general form of any element of the complex number system is: $x+yi$, where x and y are real numbers. We then call any such $x+yi$ a *complex number*.

Let z denote the complex number $x+yi$, where x and y are real numbers. We define x as the *real part* of z and y as the *imaginary part* of z. This, we indicate by writing: $x=\Re z$, and $y=\Im z$. Notice that the imaginary part is the coefficient of i in $x+yi$ and is therefore real. The fact that the word "imaginary" is used in the term "imaginary part" does not mean that it is imaginary; much the same as a human head (the head of a human being) is not a human being, but a head. (This comment is made because some books define the imaginary part to be yi (see, e.g., the Advanced Calculus of Woods, Ginn & Co.), while most books and articles use $\Im z = y$.)

If <u>z</u> denotes $x+yi$ (which we indicate by writing: $z=x+yi$), we define the <u>conjugate complex number</u> of <u>z</u> to be the complex number $x-yi$. Denoting the conjugate of <u>z</u> by $\bar{z}$, we then have: $\bar{z} = x-yi$. Therefore, $\mathcal{R}\bar{z} = \mathcal{R}z$ and $\mathcal{J}\bar{z} = -\mathcal{J}z$; we also have: $\bar{\bar{z}} = \overline{(\bar{z})} = z$.

<u>Problem 10</u>. Show that: $x=\mathcal{R}z=\frac{1}{2}(z+\bar{z})$ and $y=\mathcal{J}z = \frac{1}{2i}(z-\bar{z})$. Then show that <u>z</u> is real if and only if: $z=\bar{z}$.

<u>Problem 11</u>. Show that: $\overline{z_1+z_2}= \bar{z}_1+\bar{z}_2$ and $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$. Then, by mathematical induction, show that:

$$\overline{z_1+z_2+\ldots+z_n} = \bar{z}_1+\bar{z}_2+\ldots+\bar{z}_n \quad \& \quad \overline{z_1 z_2\ldots z_n}=\bar{z}_1\bar{z}_2\ldots\bar{z}_n.$$

The <u>absolute value</u>, or <u>modulus</u> (or <u>length</u>), of $z=x+yi$, which is denoted by $|z|$ or $|x+yi|$, is defined to be the (positive) signed square root of x^2+y^2, that is: $|z| = |x+yi| = +\sqrt{x^2+y^2}$, and is so a non-negative real number. Thus we have, e.g., $|x+0i| = |x|$, $|z| = |-z| = |\bar{z}|$, $|i| =1$, $|yi| = |y|$, $|1+i|=\sqrt{2}$.

<u>Problem 12</u>. Show that: $z\bar{z}= |z|^2= |\bar{z}|^2 \geqq 0$. Also show that: $|z|= 0$ if and only if $z=0$. Hence infer that every $z\neq 0$ has an <u>inverse</u>, z^{-1}, given by: $z^{-1}= \bar{z}/|z|^2 = 1/z = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$, where $z=x+yi$, which satisfies the equations: $z^{-1}z = zz^{-1}= 1$, and is unique (see Prob. 7). From this show that: $|z^{-1}| = 1/|z|$, and that: $1/\bar{z} = \overline{1/z}$, and that: $z'/z = z^{-1}z' = z'z^{-1}$.

<u>Problem 13</u>. Show that: $i^{-1}=1/i=-i=\bar{i}= -1/\bar{i}$.

<u>Problem 14</u>. By writing $\sqrt{z_1\bar{z}_1 z_2\bar{z}_2} = \sqrt{z_1\bar{z}_1}\sqrt{z_2\bar{z}_2}$, show that $|z_1 z_2| = |z_1||z_2|$; then by mathematical induction show that:

$$|z_1 z_2\ldots z_n| = |z_1||z_2|\ldots|z_n|.$$

<u>Problem 15</u>. Prove that $\mathcal{R}z \leqq |z|$. From this show that:

$$|z+z'|^2 =(z+z')\overline{(z+z')}=(z+z')(\bar{z}+\bar{z}')= |z|^2 + |z'|^2 + 2\mathcal{R}(z\bar{z})$$
$$\leqq |z|^2+ |z'|^2 + 2|z||z'| = (|z| + |z'|)^2.$$

From this infer that $|z+z'| \leqq |z| + |z'|$, and then by mathematical induction show that: $|z_1+z_2+\ldots+z_n| \leqq |z_1| + |z_2| +\ldots+ |z_n|$. (This result is called the <u>triangular inequality</u>.)

<u>Problem 16</u>. Prove that: $|z_1-z_2|= 0$ if and only if $z_1=z_2$, and that: $|z_1-z_2|= |z_2-z_1|$ and $|z_1-z_2| + |z_2-z_3| \geqq |z_1-z_3|$.

<u>Problem 17</u>. Show that if $z=x+yi$ then:

$$iz= -y+xi, \quad -iz = y-xi, \quad -\overline{iz} = y+xi = i\bar{z}.$$

§ 2. THE ELEMENTARY FUNCTIONS AND EULER'S FORMULAE.

The sequence $z_1, z_2, z_3, \dots$ of complex numbers z_n is said to converge to a complex number z, if and only if for each $\varepsilon > 0$ a corresponding integer N always exists such that for all $n > N$ we have $|z - z_n| < \varepsilon$. The sequence is then said to be convergent, and the complex number z is called the limit of the (sequence of complex numbers) z_n as n increases indefinitely. This is expressed by writing:

$$\lim_{n\to\infty} z_n = z, \qquad z_n \to z \ (\text{as } n \to \infty), \quad \text{or simply} \quad \lim z_n = z.$$

When the sequence is convergent, its limit must be unique; because if z' is also a limit of the sequence, an integer N' exists such that $n > N'$ implies $|z' - z_n| < \varepsilon$; but by Problem 15 of §1 we have:

$$|z - z'| = |z - z_n + z_n - z'| \leq |z - z_n| + |z_n - z'| < \varepsilon + \varepsilon = 2\varepsilon$$

for all n greater than N and N', and so the difference $z - z'$ in absolute value is less than every positive number; which can only be so if it is zero; and this in turn means that $z = z'$ (by Prob. 16, §1).

The fundamental result on the convergence of sequences is Cauchy's general convergence principle: The sequence $z_1, z_2, z_3, \dots$ shall be convergent if and only if for each $\varepsilon > 0$ a corresponding integer N always exists such that $|z_{N+n} - z_N| < \varepsilon$ for all positive integers n. (For a proof of this important result we refer the reader to K. Knopp's Elements of the Theory of Funtions, N.Y., 1952, p. 74.)

Now, for each positive integer n, let us compute the sum:

$$s_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \dots + z_n \quad \text{(called the partial sums of the sequence of the } z_n\text{).}$$

When the sequence $s_1, s_2, s_3, \dots$ converges (to s, say), we say that the series $z_1 + z_2 + z_3 + \dots$, denoted also by $\sum_{n=1}^{\infty} z_n$ or simply $\sum_n z_n$, is convergent to the value (or sum) s; and this is expressed by writing:

$$z = \lim_{m\to\infty} \sum_{n=1}^{m} z_n = \sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \dots \quad \text{or simply} \quad \sum_n z_n = s.$$

The fundamental result on the convergence of series is Cauchy's general convergence criterion: The series $z_1 + z_2 + z_3 + \dots$ shall be

(cont'd, next)

convergent if and only if for each $\varepsilon>0$ a corresponding integer N always exists such that $|z_{N+1}+z_{N+2}+\dots+z_{N+n}|<\varepsilon$ for all positive integers n. The proof of this is immediate from Cauchy's general convergence principle for sequences; because the s_n shall converge if and only if for each $\varepsilon>0$ a corresponding integer N exists such that $|s_{N+n}-s_N| = |z_{N+1}+z_{N+2}+\dots+z_{N+n}|<\varepsilon$ for all positive integers n. In particular, if the series converges then $|z_n|\to 0$ (and so $z_n\to 0$) as $n\to\infty$.

There are several very simple and useful sufficient conditions for the convergence of series. For example, the Gauss-Cauchy comparison test: If $a_1+a_2+a_3+\dots$ is a convergent series of non-negative terms a_n and $|z_n|\leqq a_n$ for all n greater than some positive integer N', then the series $z_1+z_2+z_3+\dots$ is convergent. For if $\varepsilon>0$, $|z_{N+1}+\dots+z_{N+n}|\leqq|z_{N+1}|+\dots+|z_{N+n}|\leqq a_{N+1}+\dots+a_{N+n}<\varepsilon$ for some $N>N'$ (which exists because the series of the a_n is convergent) and all n.

Problem 1. Show that the geometric series $a+ar+ar^2+ar^3+\dots$ (where a and r are real numbers) is convergent for $|r|<1$.

Another such example is D'Alembert's ratio test: If $\left|\frac{z_{n+1}}{z_n}\right|\leqq r<1$ (where r is independent of n) for all n greater than some positive integer m, then the series $z_1+z_2+z_3+\dots$ is convergent. For if $\varepsilon>0$, $|z_{N+1}+\dots+z_{N+n}| = |z_{N+1}|+\dots+|z_{N+n}|\leqq|z_m|r^{-m}(r^{N+1}+\dots r^{N+n})<|z_m|r^{-m}\varepsilon'=\varepsilon$ for some $N>m$ and all n, since the geometric series converges.

Let now n be a non-negative integer. The n^{th} power of a complex number z, which is denoted z^n, can be defined by the following recursive scheme: $z^0=1$ & $z^{n+1}=z\cdot z^n$. When n is a negative integer and $z\neq 0$, we define z^n to be $1/z^{-n}$.

Problem 2. If n is a positive integer, show by complete induction on n that $(z_1z_2)^n = z_1^n z_2^n$, and then show that $z^{-n} = (z^{-1})^n$ if $z\neq 0$. Finally show that for all integers m and n we have:

$$z^m\cdot z^n = z^{m+n} \qquad \text{and} \qquad (z^m)^n = z^{mn}$$

(assuming that $z\neq 0$ if anyone of the exponents is negative).

(cont'd on p. 87)

For any complex number $z=x+yi$ we can now form all the positive powers, z^n $(n=1,2,\dots)$, and then the following sums (where $n!=1\times 2\times\dots\times n$)

$$s_n = 1 + z + z^2/2! + \dots + z^n/n! . \qquad (n=1,2,\dots)$$

It is not difficult to show that these s_n always converge (e.g., by D'Alembert's ratio test), for each z, to a corresponding (unique) limit which is denoted: exp z; that is, the function whose values are:

$$\exp z = 1 + z + z^2/2! + z^3/3! + \dots = \sum_{n=0}^{\infty} z^n/n! = \lim_{n\to\infty}\sum_{m=0}^{n} z^m/m!, \quad (1)$$

is always defined for every complex number z. The number exp 1 is universally denoted by the letter e, and when z is an integer exp z reduces to e^z, so that exp z is frequently written e^z, for all z.

In like manner, it can be shown that the following sums:

$$\sum_{n=0}^{n}(-1)^n\frac{z^{2n+1}}{(2n+1)!} = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots+(-1)^n\frac{z^{2n+1}}{(2n+1)!} \quad \& \quad \sum_{n=0}^{n}(-1)^n\frac{z^{2n}}{(2n)!} = 1-\frac{z^2}{2!}+\dots+(-1)^n\frac{z^{2n}}{(2n)!},$$

always converge for each complex number z. When z is a real number (i.e. when $\mathcal{I}z=0$) these sums converge to the familiar sine and cosine of z, respectively; and so in general we define:

$$\sin z = \sum_{n=0}^{\infty}(-1)^n z^{2n+1}/(2n+1)! = z-z^3/3!+z^5/5!-z^7/7!+\dots, \qquad (2)$$

$$\cos z = \sum_{n=0}^{\infty}(-1)^n z^{2n}/(2n)! = 1-z^2/2!+z^4/4!-z^6/6!+z^8/8!+\dots \qquad (3)$$

It can be shown that the terms in eq. (1) can be rearranged thus:

$$e^z = \exp z = (1+z^2/2!+z^4/4!+\dots) + (z+z^3/3!+z^5/5!+\dots).$$

Consequently, substituting zi for z and noting that: $i^{2n}=(i^2)^n=(-1)^n$ and $i^{2n+1}=i^{2n}i=(-1)^n i$, we infer, from eq. (2 & 3), that:

$$e^{zi} = \exp(zi) = \cos z + i.\sin z, \qquad (4)$$

which is *Euler's formula* for any complex number z.

It is customary to denote e^{zi} and $(\cos z + i \sin z)$ by cis z, or $\angle z$, although the latter is usually reserved for real z. (Both are read: "cis" z; "cis" comes from cosine-i-sine.)

Problem 3. Show that: $e^{-zi} = \cos z - i.\sin z$ & $e^z=\cos(zi)-i\cdot\sin(zi)$. These are other forms of Euler's formula.

Problem 4. Show that we have the following equations:

$\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\sin z = \frac{1}{2i}(e^{zi}-e^{-zi})$, $\cos z = \frac{1}{2}(e^{zi}+e^{-zi})$.

The last two formulas are called Euler's formulas for the sine and cosine, respectively.

Problem 5. Show that: $e^{z_1+z_2} = e^{z_1}e^{z_2}$; then be mathematical induction show that: $e^{z_1+z_2+\dots+z_n} = e^{z_1}e^{z_2}\dots e^{z_n}$; and thence: $(e^z)^n = e^{nz}$ for every positive integer n.

Problem 6. Show that: $\sin^2 z + \cos^2 z = (\sin z)^2 + (\cos z)^2 = 1$,

and: $\sin(z'+z'') = \sin z'.\cos z'' + \sin z''.\cos z'$,
$\cos(z'+z'') = \cos z'.\cos z'' - \sin z'.\sin z''$.

Problem 7. Show that $\exp 0 = 1$. Hence infer that: $e^{-z} = 1/e^z$, and that: $(e^z)^n = e^{nz}$, for every integer n (positive or negative, or 0).

Problem 8. Show that for any integer n we have:
$\sin(z+2n\pi) = \sin z$, $\cos(z+2n\pi) = \cos z$, $\exp(z+2n\pi i) = \exp z$.
Consequently, sin z and cos z are periodic with real periods $2n\pi$, and exp z is periodic with the imaginary periods $2n\pi i$, for all integers $n \neq 0$. It can be shown also (see Whittaker & Watson, Modern Analysis, Cambridge University Press, 1927, Appendix) that these are the only periods; that is, if any complex number $\overset{a\neq0}{\wedge}$ is a period of sin z or cos z, then there must be an integer $n\neq0$ such that: $a=2n\pi$; and if $a\neq0$ is a period of exp z, then $a=2n\pi i$ for some integer $n\neq0$.

Problem 9. Show that for every real number ϑ, we have:

$$|e^{\vartheta i}| = 1, \quad \overline{e^{\vartheta i}} = e^{-\vartheta i} = 1/e^{\vartheta i}.$$

In other symbols, these become: $|\underline{/\vartheta}| = 1$, $\overline{\underline{/\vartheta}} = \underline{/-\vartheta} = \overline{\text{cis}\,\vartheta} = \text{cis}(-\vartheta) = \frac{1}{\underline{/\vartheta}} = \frac{1}{\text{cis}\,\vartheta}$.
The quantity $\underline{/-\vartheta}$ is sometimes denoted by $\overline{\backslash\vartheta}$; but this notation is clumsy and unnecessary and will not be used here.

Problem 10. Show that for every integer n we have:

$$e^{\pm2n\pi i} = \underline{/\pm2\pi n} = 1, \quad e^{\pm(2n+1)\pi i} = \underline{/\pm(2n+1)\pi} = -1, \quad e^{(4n\pm1)\frac{\pi}{2}i} = \underline{/(4n\pm1)\tfrac{\pi}{2}} = \pm i = \mp\bar{i} = \frac{\mp1}{i}$$

$$\underline{/\alpha+\beta} = \underline{/\alpha}\cdot\underline{/\beta}, \quad \frac{\underline{/\alpha_1}\,\underline{/\alpha_2}\cdots\underline{/\alpha_m}}{\underline{/\beta_1}\,\underline{/\beta_2}\cdots\underline{/\beta_n}} = \underline{/\alpha_1+\alpha_2+\dots+\alpha_m-(\beta_1+\beta_2+\dots+\beta_n)}.$$

Problem 11. Show that if $z=x+yi$ (x & y = real numbers), then: $e^z = e^x(\cos y + i.\sin y) = e^x\underline{/y}$, and hence: $|e^z| = e^x$.

Problem 12. The tangent, cotangent, secant, and cosecant, of any complex number z are defined, as usual for real numbers, as follows (when the denominators are not zero):

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = 1/\cos z, \quad \csc z = 1/\sin z.$$

Show that we have:

$$\tan z = -i\frac{e^{zi}-e^{-zi}}{e^{zi}+e^{-zi}}\ , \quad \cot z = i\,\frac{e^{zi}+e^{-zi}}{e^{zi}-e^{-zi}}\ , \quad \sec z = \frac{2}{e^{zi}+e^{-zi}}$$

$$\csc z = \frac{2i}{e^{zi}-e^{-zi}}\ , \quad \sec^2 z = 1 + \tan^2 z\ , \quad \csc^2 z = 1 + \cot^2 z.$$

Problem 13. The hyperbolic functions are defined as follows:

$$\sinh z = \frac{1}{2}\,(e^{z}-e^{-z}), \quad \cosh z = \frac{1}{2}(e^{z}+e^{-z}), \quad \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}\ ,$$

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{e^{z}+e^{-z}}{e^{z}-e^{-z}}\ , \quad \operatorname{sech} z = 1/\cosh z, \quad \operatorname{csch} z = 1/\sinh z.$$

Show that we have: $e^{\pm z} = \pm\sinh z + \cosh z$, $\sinh(-z)=-\sinh z$,

$\cosh(-z) = \cosh z$, $\tanh(-z)=-\tanh z$, $\coth(-z)=-\coth z$,

$\sinh(zi)=i.\sin z$, $\cosh(zi)=\cos z$, $\tanh(zi)=i.\tan z$, $\coth(zi)=-i.\cot z$,

$\sin(zi)=i.\sinh z$, $\cos(zi)=\cosh z$, $\tan(zi)=i.\tanh z$, $\cot(zi)=-i.\coth z$,

$\cosh^2 z - \sinh^2 z = 1$, $\operatorname{sech}^2 z = 1-\tanh^2 z$, $\operatorname{csch}^2 z = \coth^2 z - 1$,

$$\sinh(z'\pm z'') = \sinh z'.\cosh z'' \pm \sinh z''.\cosh z'$$

$$\cosh(z'\pm z'') = \cosh z'.\cosh z'' \pm \sinh z'.\sinh z''.$$

Show also that $2n\pi i$ (n=integer$\neq 0$) is a period of all the hyperbolic functions.

§3. GRAPHICAL REPRESENTATIONS AND GAUSS-ARGAND DIAGRAMS.

Given any complex number z, the real part x and the imaginary part y are uniquely determined. We can then construct a point with the rectangular cartesian coordinates (x,y) on any given plane, as soon as rectangular cartesian axes are chosen on that plane. In this way there shall correspond to each complex number z, a unique point (x,y) of the plane. Moreover, given any point of the plane, the abscissa and ordinate are uniquely determined (with respect to the axes chosen) and so there shall be determined a unique complex number with the abscissa as real part and with the ordinate as imaginary part. In this way there is established a one to one correspondence between the class of complex numbers on the one hand and the set of points of the given plane on the other. The set of real numbers will correspond to the axis of the abscissae and so this line will be called the real (number) line, or the real axis too. The set of the imaginary numbers of the form $yi=0+yi$ (y=real) will correspond to the axis of the ordinates and so this line will be called the imaginary line, or imaginary axis also.

We will take the liberty of identifying a complex number with the corresponding point on the given plane, and we will often speak of a point on that plane, meaning the corresponding complex number. Thus we will speak of addition, subtraction, multiplication, division, etc. of points on the given plane, meaning the corresponding operations on the corresponding complex numbers. For this reason, we speak of the complex number system as the (open) complex plane. If we augment this system with the point at infinity, ∞, to which we assume converge all sequences of complex numbers with moduli steadily increasing beyond all bounds, we get what is called the closed complex plane. By means of a stereographic projection, a one to one correspondence between the points of the closed plane and the points of (the surface of) a sphere with radius > 0, tangent to the plane (e.g. at the origin of a chosen set of axes), can also be established. For this reason we also speak of the system of complex numbers augmented with the point at infinity as the complex sphere.

Since a point on the complex plane determines a unique directed line segment (issuing) from the origin to that point and, viceversa, a directed line segment from the origin determines a unique terminal point, there will be established a one to one correspondence between the points of the complex plane (and therefore between the complex numbers) and the directed line segments issuing from the origin. Moreover, a directed line segment issuing from the origin determines uniquely a class of (the totality of) parallel directed line segments of equal magnitude and sense, called a vector (although this word is usually reserved for any generic directed line segments of the class), and viceversa.

Therefore a one to one correspondence will be established in this way between the directed line segments issuing from the origin (and therefore also between the complex numbers) and the set of vectors. On account of this we will also take the liberty of identifying a complex number and a directed line segment with the corresponding vector in the complex plane, and we will frequently use one for the other indistinctly.

Considering again an arbitrary complex number $z=x+yi$ (x and y real), let us make the following substitution (change of variables):

$$x = r.\cos\vartheta \qquad y = r.\sin\vartheta\ . \tag{1}$$

Taking x and y as rectangular cartesian coordinates of a point in a plane, the new variables r and ϑ can be recognized as the polar coordinates of this point, in terms of which the complex number z becomes:

$$z = r(\cos\vartheta + i\sin\vartheta) = r\,e^{\vartheta i} = r\,\mathrm{cis}\,\vartheta = r\angle\vartheta\ , \tag{2}$$

by eq. (4). These forms of expressing the complex number z are called its polar forms because they are given in terms of the polar coordinates of (the point corresponding to) z (and more specifically, the first form is called the trigonometric form, the second is called the exponential form, and the last two forms are called the conventional or symbolic forms, of the complex number z); the earlier form, namely, $x+yi$, is called the rectangular form of (expressing the complex number) z; because it is expressed in terms of the rectangular coordinates of z.

It is easy to see that there always exist numbers r and ϑ which satisfy eq. (1), for any given real numbers x and y. In fact, by Prob. 6, §2 (p. 88), we have:

$$x^2 + y^2 = (r.\cos\vartheta)^2 + (r.\sin\vartheta)^2 = r^2(\cos^2\vartheta + \sin^2\vartheta) = r^2; \tag{3}$$

and by Prob. 12, §2, we have, if $x \neq 0$:

$$y/x = r.\sin\vartheta/(r.\cos\vartheta) = \sin\vartheta/\cos\vartheta = \tan\vartheta\ , \tag{4}$$

and if $x=0$, we get directly (assuming $r\neq 0$) that $\cos\vartheta = 0$ and so that: $\vartheta = \pm\frac{\pi}{2}$, or any multiple thereof. (When $x=r=0$, ϑ is arbitrary and irrelevant.)

From eq. (3), we find that $r=\pm|z|$, and from eq. (4) we find an infinity of values of ϑ, which, however, differ by integral multiples of 2π (radians, or 360°) once one of the values: $\pm|z|$, is chosen for r. We will always choose $r \geqq 0$, so that we will have:

$$r = |z| = +\sqrt{x^2+y^2} = +\sqrt{(\mathcal{R}z)^2+(\mathcal{J}z)^2}; \tag{5}$$

then any value of $\vartheta = \tan^{-1}(y/x)$ which together with $r=+|z|$ satisfies eq. (1) will be called an angle, amplitude, or argument, of the complex number z, a situation which will be indicated by writing: $\vartheta = \text{ang}\, z$; hence:

$$\text{ang}\, z = \tan^{-1}(y/x) = \tan^{-1}(\mathcal{J}z/\mathcal{R}z), \tag{6}$$

together with the condition that the cosine and sine of this angle shall have the same signs as x and y, respectively. (When the symbolic notation is used, the angles are usually expressed in degrees.)

The unique angle of z, i.e. the unique value of the infinitely many-valued function: ang z, satisfying the inequalities:

$$-\pi < \text{ang}\, z \leqq +\pi \tag{7}$$

is called the principal value of the angle, or phase, of z, i.e., the principal value of the function: ang z. This principal value is denoted by: Ang z (with a capital A), and for each value of ang z there shall always be an integer n such that: $\text{ang}\, z = \text{Ang}\, z + 2n\pi = \text{Ang}\, z + 360°$.

When two numbers a & b differ by an integral multiple of 2π they are said to be congruent modulo 2π, and this situation is indicated by writing $a \equiv b \pmod{2\pi}$. Accordingly, all the angles of a complex number z are congruent modulo 2π, and we have: $\text{ang}\, z = \text{Ang}\, z \pmod{2\pi}$.

Problem 1. If $z = r\angle\vartheta$, show that: $zi = r\angle\vartheta+90°$ & $-zi = r\angle\vartheta-90°$; that is: $\text{ang}(zi) = \text{ang}(z) + 90°$ and $\text{ang}(-zi) = \text{ang}(z) - 90°$.

Problem 2. If $z_n = r_n\angle\vartheta_n$ $(n=1,2,\dots,N)$, $z'_m = r'_m\angle\vartheta'_m \neq 0$ $(m=1,2,\dots,M)$ show that we have:

$$\frac{z_1\cdot z_2\cdots z_N}{z'_1\cdot z'_2\cdots z'_M} = \frac{r_1\cdot r_2\cdots r_N}{r'_1\cdot r'_2\cdots r'_M}\angle\vartheta_1+\vartheta_2+\cdots+\vartheta_N-(\vartheta'_1+\vartheta'_2+\cdots+\vartheta'_M).$$

Then, from this, infer that:

$$\left|\frac{z_1\cdot z_2\cdots z_N}{z'_1\cdot z'_2\cdots z'_M}\right| = \frac{|z_1|\cdot|z_2|\cdots|z_N|}{|z'_1|\cdot|z'_2|\cdots|z'_M|},$$

and that:

$$\text{ang}\,\frac{z_1\cdot z_2\cdots z_N}{z'_1\, z'_2\cdots z'_M} = \text{ang}\, z_1+\cdots+\text{ang}\, z_N-(\text{ang}\, z'_1+\cdots+\text{ang}\, z'_M);$$

this last equation should be understood to mean that there are values (among the many values) of ang z_n and ang z'_m which satisfy this equation; but this is not necessarily true with Ang z_n and Ang z'_m, unless the relation is considered as a congruence modulo 2π.

Problem 3. Suppose z is any complex number $\neq 0$; its natural logarithm, denoted $\ln z$, is any complex number w (say) such that $\exp w = e^w = z$. Since the exponential function, exp, is periodic (cf. Prob. 8, §2), the natural logarithm function shall be many-valued. Show that if w_0 is any one value of $\ln z$, (all) the other values of $\ln z$ are given by the expression: $w_0+2n\pi i$ $(n=\pm 1, \pm 2, \ldots)$. Show also that $\ln z = \ln|z| + i \operatorname{ang} z$ (for $z \neq 0$), in which $\ln|z|$ may be (and usually is) taken as the unique real natural logarithm of the positive number $|z|$ (as found in tables). The principal value of $\ln z$ is (defined to be:) $\operatorname{Ln} z = \ln z + i \operatorname{Ang} z$.

Problem 4. If $zz' \neq 0$, show that: $\ln z + \ln z' = \ln(zz')$ and then, by mathematical induction, show that if $z_1 z_2 \ldots z_N \neq 0$, we shall have:

$$\ln(z_1 z_2 \ldots z_N) = \ln z_1 + \ln z_2 + \ldots + \ln z_N,$$

in the sense that there are values (among the many values) of each term in this equation which satisfy the equation; but one must note that this equation is not necessarily true with principal values, unless it is considered as a congruence modulo $2\pi i$.

Problem 5. The general a^{th} power, z^a, of $z \neq 0$ is defined for all complex exponents to be $\exp(a \ln z)$. Unless a is an integer, z^a is a many-valued function of z. (When a is a positive (real) number, z^a is defined to be 0 for $z=0$.) Show that we have:

$$z^a z^b = z^{a+b}, \quad (z^a)^b = z^{ab}, \text{ and if } w = z^a \text{ then } z = w^{1/a}.$$

The principal value of z^a is defined to be $\exp(a \operatorname{Ln} z)$. (Note: The general exponential function to the base $a \neq 0$ is defined for all complex numbers z to be $a^z = \exp(z \ln a)$, once a particular value of $\ln a$ is chosen; then, for this choice, the logarithms to the base a are defined by the statement: $w = \log_a z$ if and only if $z = a^w$.)

Problem 6. The inverse trigonometric functions are many-valued. Express $\sin^{-1} w$, $\cos^{-1} w$, etc., as logarithms. (Hint: If e.g., $w = \sin z = (e^{iz}-e^{-iz})/2i$, then $e^{2iz}-2iwe^{iz}-1=0 \therefore z = -i \ln(iw \pm \sqrt{1-w^2})$, etc.)

Problem 7. The inverse hyperbolic functions are many-valued. Express: $\sinh^{-1} z$, $\cosh^{-1} z$, etc., as logarithms. (Hint: If $w = \sinh^{-1} z$, $z = \sinh w = (e^w - e^{-w})/2 \therefore (e^w)^2 - 2z\, e^w - 1 = 0 \therefore e^w = z \pm \sqrt{z^2+1}$; etc.)

The importance of the one to one correspondences considered above lies in the possibilities they afford of representing and visualizing complex numbers and operations with them from a geometrical or graphical point of view, by means of diagrams on a (complex number) plane known as Gauss-Argand diagrams, due to J.F.K. Gauss (1797), C. Wessel (1797), and (in their present form to) J.R. Argand (1806).

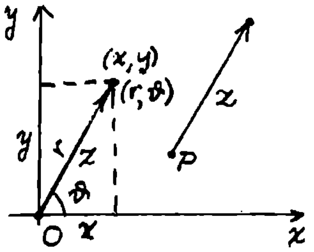


Fig.1.

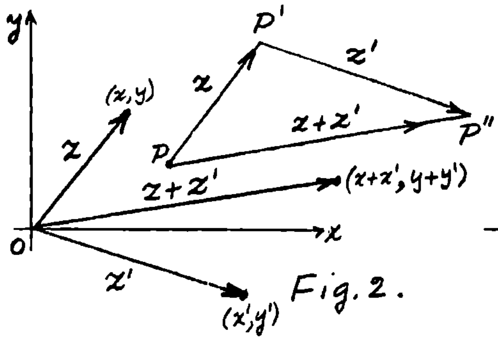


Fig.2.

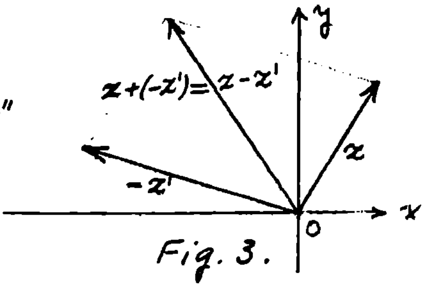


Fig.3.

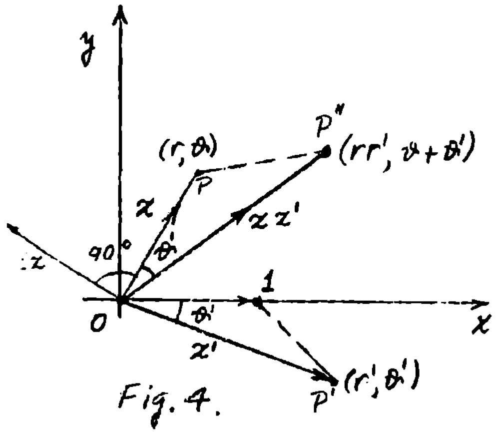


Fig.4.

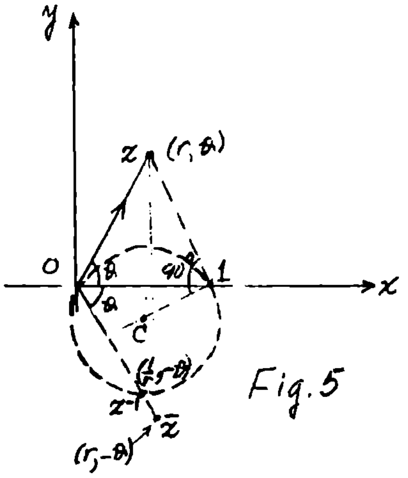


Fig.5

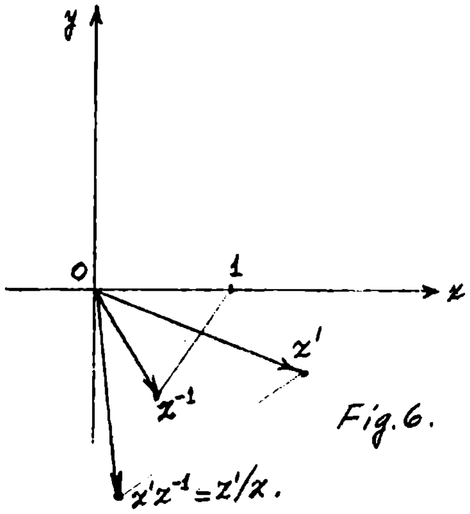


Fig.6.

In Fig. 1, the complex number z=x+yi is shown as a point with the rectangular cartesian coordinates (x,y) and with the polar coordinates (r,ϑ), and also as a directed line segment from the origin to that point, and as a generic parallel directed line segment of equal size and sense issuing from a generic point P, representing the vector z. In Fig. 2, the addition of two complex numbers z and z' is performed according to the well known parallelogram law, which consists in (algebraically) adding corresponding rectangular cartesian coordinates in order to obtain those of the sum; the construction is shown, first, by considering the complex numbers as directed line segments issuing from the origin and, second, by drawing z from an arbitrary point P to the point P' and then z' from P' to P'', the sum being represented by the directed line segment from P to P''. In Fig. 3, the subtraction of z' from z is shown as the addition of $-z'$ to z. In Fig. 4, the multiplication of z by z' is shown; this operation consists in multiplying the moduli, and adding the angles, of z and z' in order to obtain the polar coordinates of the product; graphically the operation is performed by drawing z and z' at a point taken as origin and then the vector corresponding to the real number $1=1\angle 0$, and then the triangle OPP'' similar to the triangle $O1P'$. In Fig. 5, the inverse, z^{-1}, of z is obtained by the intersection of the line from O to the conjugate, $\bar{z}$, of z with the circumference through the points $O=0\angle 0$ and $1=1\angle 0$ and with its center on the line $\overline{1C}$, perpendicular to the line joining the unit point 1 to the point z. In Fig. 6, the division of z' by z is shown as the multiplication of z' by z^{-1}. (Fig. 4 is used to show the multiplication of a complex number z by the imaginary unit i; this operation consists simply in rotating z by a right angle in the positive direction, i.e. counterclockwisely; multiplication by $-i=\angle -90°$ would be performed by a rotation of a right angle in the negative direction, i.e. clockwisely; and multiplication by a complex number of unit modulus, of the form $e^{\vartheta i}=\angle\vartheta$, where ϑ is real, would be performed by a rotation of an angle ϑ.) (ϑ radians $=\frac{\vartheta}{\pi}\times 180°$)

For many purposes it is convenient to perform the transformation indicated by eq. (1) in order to transform a complex number from its rectangular form to its polar form, and viceversa. This, of course, is the same familiar problem of transforming rectangular coordinates to polar coordinates, in a plane, and viceversa.

The practical way of transforming the rectangular $z=x+yi$ to the polar form $r\angle\vartheta$ is as follows. Divide the absolute value of the imaginary part by that of the real part (assuming the latter to be $\neq 0$; for if it were 0 the transformation would be trivial). With the use of tables of tangents (or of a slide rule), the value of $\tan^{-1}|y/x|=\varphi$ (say) can be found, and then $\sin\varphi$ can be found from a table of sines (or from the slide rule). With this we get the modulus: $|z|=|y|/\sin\varphi$. Finally, $\vartheta=$ Ang z can be found simply from φ by paying attention to the quadrant in which the point (x,y) lies. Thus, if $x>0$ and $y\geqq 0$, then $\vartheta=\varphi$, and if $x>0$ and $y<0$, then $\vartheta=-\varphi$. Also, if $x<0$ and $y\geqq 0$, then $\vartheta=180°-\varphi$, and if $x<0$ and $y<0$, then $\vartheta=-(180°-\varphi)$. When $x=0$ and $y>0$, $r=y$ and $\vartheta=90°$; and when $x=0$ and $y<0$, $r=|y|$ and $\vartheta=-90°$.

This method has one inconvenience: it is not very accurate when $x=\Re z$ is small, i.e. when φ is near 90°. To avoid this it is convenient to use the quotient of the smaller of $|x|$ and $|y|$ divided by the larger, instead of using the quotient $|y/x|$, in the manner used by J.C.P. Miller in his Tables for converting rectangular to polar co-ordinates (Scientific Computing Service, Ltd., London). Let k denote the quotient of the smaller of $|x|$ and $|y|$ by the larger. This quotient is always $\leqq 1$ and $\tan^{-1}k=\vartheta_0$ (say) can be found very accurately. Then the modulus $|z|$ can be found by multiplying the larger of $|x|$ and $|y|$ by $(1+k^2)^{1/2}=\sec\vartheta_0$ (which can be read off the tables), or by dividing by $\cos\vartheta_0$ (which can be found from a slide rule). [Actually, it is better to compute $|z|$ by the following formula: $|z|-\max(|x|,|y|)=(\sqrt{1+k^2}-1)\times\max(|x|,|y|)$.] Finally, $\vartheta=$ Ang z can be found from ϑ_0 by finding the angle ϑ so as to be in the same octant, into which the bisectrices divide the quadrants, as the point (x,y); as shown in Fig. 7. Thus, if $x>0$, $y\geqq 0$, and $y\leqq x$, then $\vartheta=\vartheta_0$; but if $x>0$, $y>0$, and $x\leqq y$, then $\vartheta=90°-\vartheta_0$; etc.

The transformation from the polar form, $r\angle\vartheta$, to the rectangular form, $x+yi$, offers no difficulty whatever. First, we obtain $\sin\vartheta$ and $\cos\vartheta$ from tables, or from a slide rule. Then we obtain the real part x and the imaginary part y, of the complex number, by: $x=r\cos\vartheta$ and $y=r\sin\vartheta$

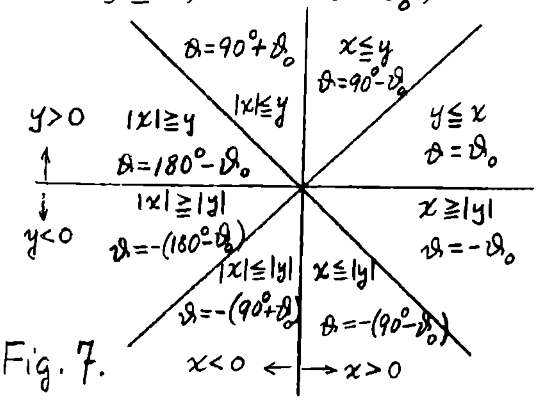


Fig. 7.

§4. SINE FUNCTIONS (SINUSOIDS).

The function f of a variable t given by: $f(t) = A.\sin(\omega t+\alpha)$, where A, $\omega(\neq 0)$, and α are constant, is called a sine function, or a sinusoid. This function has the periods $2n\pi/\omega$ for all integers $n\neq 0$. The smallest positive period, $2\pi/|\omega| = T$ (say), is called the fundamental period, or simply the period. All other periods are multiples of this fundamental period. The number of fundamental periods in a unit of t is called the frequency, ν (say), of the sine function, and is given by: $\nu = 1/T = 1/(2\pi/|\omega|) = |\omega|/2\pi$. The quantity $|\omega| = 2\pi\nu = 2\pi/T$ is called the 2π-frequency, the angular frequency, the circular frequency (especially in mechanics), and sometimes the radian frequency. Strictly, both the frequency ν and the 2π-frequency $|\omega|$ should be measured in the inverse unit of t; but it is customary to give a frequency in cycles per unit of t, and a 2π-frequency in radians per unit of t, in order to indicate that one is referring to a frequency, and a 2π-frequency (respectively). When t is the time and the unit of t is the second (as in the MKSC-system of units), the unit of frequency is taken as the cycle per second (abbreviat~~ed: cps~~, c/s, or ν), also called a Hertz (abbreviated Hz.), and the unit of 2π-frequency is taken as a radian per second (abbreviated: rad/sec), although both are strictly the inverse second (sec^{-1})

The quantity $|A|$ is called the amplitude of the sine function, and A itself may be called its coefficient; both have the same unit as the sine function. The quantity α, is called the (initial) angle of the sine function and is, of course, dimensionless (as well as the product: ωt).

Since we have that:

$$A.\sin(-\omega t+\alpha) = -A.\sin(\omega t-\alpha) = A.\sin(\omega t-\alpha\pm 180°),$$

and since we can always add (or subtract) any integral multiple of 2π radians (i.e. 360°) without altering the values of a sinusoid, any sine function $A.\sin(\omega t+\alpha)$ can always be reduced to an equal sine function $A'.\sin(\omega' t+\alpha')$, where $A' = |A| \geqq 0$, $\omega' = |\omega| \geqq 0$, and $-\pi < \alpha' \leqq +\pi$, without changing the independent variable t. The angle α' is then called the principal angle, or phase, of the sine function. (In general, it is always possible to make a transformation of this kind in such a way as to bring the angle of the sine function into any given interval of length 2π, i.e., 360°.) On account of this, we can always assume, without loss of generality, that the coefficient of a (real-valued) sine function is equal to its amplitude and so is non-negative, that its 2π-frequency (if $\neq 0$) is positive, and that its angle is equal to its principal angle, or phase.

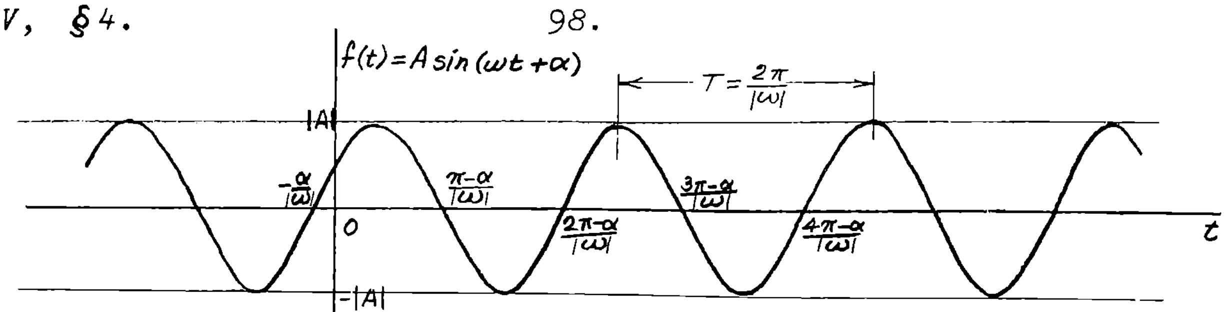


Fig.1.

In Fig. 1 is shown the graph of the sinusoid: $f(t)=A\sin(\omega t+\alpha)$ (for $A>0$ and $\omega>0$). By making the substitution: $x=\omega t+\alpha$ and $y=f/|A|$, which consists in a change of origin and of scale along the axis of the abscissæ and a change of scale along the axis of the ordinates, this graph may be considered as that of the special sine function: $\underline{\sin x}$; and by making an inverse transformation of this type the graph may be considered as that of any other sinusoid. Thus we may confine ourselves to a single graph, e.g., that of the special function $\underline{\sin x}$.

<u>Problem 1</u>. Show that in any collection of sine functions with angular frequencies $\neq 0$, a change in origin of the independent variable $\underline{t}$ (say) can always be made in such a way as to make the phase of any one sinusoid of the collection have an angle = 0. [Hint: If $\underline{A.\sin(\omega t+\alpha)}$ is the sinusoid in question, make the following substitution: $t=t'+\tau$, in terms of the new time $\underline{t'}$, such that: $\omega t+\alpha=\omega t'$.] Also show that if all the angular frequencies of the sinusoids of the collection are the same then this cannot be done simultaneously with two sinusoids with different phases or principal angles.

<u>Problem 2</u>. Show that the zeros of $\underline{A\sin(\omega t+\alpha)}$, where $\omega\neq 0$, (i.e. the values of $\underline{t}$ which annul this function) are: $t=\frac{n\pi-\alpha}{\omega}$ where $\underline{n}$ is any integer. Show also that the points of maxima and minima are given by: $\frac{(4n+1)\pi-2\alpha}{2\omega}$ and $\frac{(4n-1)\pi-2\alpha}{2\omega}$, respectively, if $A>0$ (and viceversa if $A<0$). The zeros are sometimes sub-divided into two mutually exclusive classes: $\frac{2n\pi-\alpha}{\omega}$ and $\frac{(2n+1)\pi-\alpha}{\omega}$, frequently called the 0-points and the π-points, respectively, if $A>0$ (and conversely if $A<0$). Show that at a 0-point the derivative is positive and at a π-point the derivative of the sinusoid is negative.

<u>Problem 3</u>. When the angles of two sine functions with positive coefficients, of the same frequency, differ by a multiple of 2π radians (360°) they are said to be <u>in phase</u>, and if they differ by an odd multiple of $\pi/2$ radians (90°) they are said to be in <u>quadrature</u>. By expressing any sinusoid $\underline{A\sin(\omega t+\alpha)}$ in the form $\underline{A\sin(\omega t+\beta+\alpha-\beta)}$ and then expanding the sine function (according to the second formula of Prob. 6, §2, p. 88), show that it can always be expressed as the sum of two sine functions,

one of which is in phase with $\sin(\omega t+\beta)$ and the other in quadrature with it, for any given β. Show that this decomposition is unique. (When two sinusoids with positive coefficients have angles which differ by an odd multiple of π radians (180°), i.e., when their phases differ by π radians (180°), they are said to be *180° out of phase*, or simply *out of phase*, or *in opposition*.)

Problem 4. Show that:

$$A\cos(\omega t+\alpha) = A\sin(\omega t+\alpha+\tfrac{\pi}{2}) \quad \text{and} \quad -A\sin(\omega t+\alpha) = A\sin(\omega t+\alpha\pm\pi).$$

Problem 5. Show that the sinusoid $A\sin(\omega t+\alpha)$ is the projection on the ordinate axis, and $A\cos(\omega t+\alpha)$ is the projection on the abscissa axis, of the vector $A\exp[i(\omega t+\alpha)] = Ae^{\alpha i}e^{i\omega t}$, called a *rotating vector*, since it represents a vector rotating around the origin with an angular velocity ω on a circle of radius A, originally at $Ae^{\alpha i}$.

Problem 6. Any given sinusoid $A\sin(\omega t+\alpha)$ may be expressed in the form $A'\sin\omega t + A''\cos\omega t$, where $A' = A\cos\alpha$ and $A'' = A\sin\alpha$. The sinusoid $A''\cos\omega t = A''\sin(\omega t+\frac{\pi}{2})$ is in quadrature with $A'\sin\omega t$; thus the given sinusoid is expressed as the sum of a sinusoid with zero phase angle and a sinusoid in quadrature with the latter. Show that the given sinusoid, namely $A\sin(\omega t+\alpha)$, is the sum of the projections on the ordinate axis of the rotating vectors $A'e^{i\omega t}$ and $iA''e^{i\omega t}$, the latter always being at right angles to the former.

The sum of various sinusoids of the same frequency: $A_k\sin(\omega t+\alpha_k)$ $(k=1,2,\dots,n)$, can best be obtained by expressing them in the form: $A'_k\sin\omega t + A''_k\cos\omega t$, where $A'_k = A_k\cos\alpha_k$ and $A''_k = A_k\sin\alpha_k$ $(k=1,2,\dots,n)$. In this way we get:

$$\sum_{k=1}^{n} A_k\sin(\omega t+\alpha_k) = \Big(\sum_{k=1}^{n} A'_k\Big)\sin\omega t + \Big(\sum_{k=1}^{n} A''_k\Big)\cos\omega t = A'\sin\omega t + A''\cos\omega t = A\sin(\omega t+\alpha), \tag{1}$$

where: $A' = \sum_{k=1}^{n} A'_k$, $A'' = \sum_{k=1}^{n} A''_k$, $A' = A\cos\alpha$, $A'' = A\sin\alpha$, $A = \sqrt{A'^2+A''^2}$, $\alpha = \tan^{-1}\frac{A''}{A'}$.

It can be shown that no such thing can be done with sine functions with distinct frequencies; because they are linearly independent, as shall be shown later (see §7, p. 111.)

Problem 7. Show that if: $A\sin\omega t + B\cos\omega t + C = 0$ for all t (actually much less would be sufficient), where A, B, C are constant, then $A=B=C=0$. [Hint: Substitute: $t=0$ and $t=\frac{\pi}{\omega}$ and obtain: $B+C=0$ and $-B+C=0$, from which $B=C=0$; then obtain that $A=0$.] Corollary: In an equation between sinusoids of the same frequency and constants, (which may be reduced to an equation) of the form:

$$A\sin\omega t + B\cos\omega t + C = A'\sin\omega t + B'\cos\omega t + C', \tag{2}$$

we obtain that (by passing all the terms to the first member): $A=A'$, $B=B'$, $C=C'$; hence corresponding coefficients and terms in eq. (1) are equal.

By using the well-known formula of trigonometry:

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] = \frac{1}{2}\left[\sin(x-y+\tfrac{\pi}{2}) + \sin(x+y-\tfrac{\pi}{2})\right],$$

the product of two sinusoids of 2π-frequencies ω and ω' can be expressed as a sum of sinusoids with 2π-frequencies $\omega-\omega'$ and $\omega+\omega'$:

$$A\sin(\omega t+\alpha).A'\sin(\omega' t+\alpha') = \frac{AA'}{2}\sin[(\omega-\omega')t+\alpha-\alpha'+\tfrac{\pi}{2}]+\frac{AA'}{2}\sin[(\omega+\omega')t+\alpha+\alpha'-\tfrac{\pi}{2}] \quad (3)$$

if $\omega\neq\omega'$; otherwise it can be expressed as a sum of a constant term and a sinusoid of double frequency (when $\omega=\omega'$):

$$A\sin(\omega t+\alpha)\cdot A'\sin(\omega t+\alpha') = \frac{AA'}{2}\cos(\alpha-\alpha')+\frac{AA'}{2}\sin(2\omega t+\alpha+\alpha'-\tfrac{\pi}{2}). \quad (4)$$

In any case, the product is no longer a sinusoid of 2π-frequency ω or ω'.

Problem 8. Show that the product of two sinusoids is again a sine function if and only if the two sinusoids have the same frequency and are in quadrature, in which case the product is a sinusoid of double frequency.

Problem 9. Show that if the angles of two sinusoids of the same frequency do not differ by an integral multiple of π radians (180°), then the product, which is a sinusoid $s(t)$ (say) of double frequency plus a constant c (say), is sometimes negative and sometimes positive. [Hint: Show that the amplitude of $s(t)$ is greater than $|c|$.] Concerning a circuit element with sinusoidal current and voltage, this would mean that at times the element is taking power from the circuit and at times it is returning power to the circuit; on the average it would absorb power from the circuit, given by c, sign & all.

Problem 10. If $\omega B\neq 0$, show that we have:

$$\frac{A\sin(\omega t+\alpha)}{B\sin(\omega t+\beta)} = \frac{A}{B}\cos(\alpha-\beta)+\frac{A}{B}\sin(\alpha-\beta)\cot(\omega t+\beta).$$

The derivative (with respect to t) of a given sinusoid, $A\sin(\omega t+\alpha)$, is another sinusoid of the same frequency, namely:

$$\frac{d}{dt}A\sin(\omega t+\alpha) = \omega A\cos(\omega t+\alpha) = \omega A\sin(\omega t+\alpha+\tfrac{\pi}{2}), \quad (5)$$

which is in quadrature (90° out of phase) with the given sinusoid (and in fact, leading it by $\frac{\pi}{2}$ radians, i.e., 90°).

On the other hand, the integral (with respect to t) of the given sinusoid is another sinusoid with the same frequency, namely:

$$\int A\sin(\omega t+\alpha)dt = -\frac{A}{\omega}\cos(\omega t+\alpha) = \frac{A}{\omega}\sin(\omega t+\alpha-\tfrac{\pi}{2}), \quad (6)$$

(which is also in quadrature with the given sinusoid, and, in fact, lagging it by 90°) plus an integration constant. However, in an equation between linear combinations of sinusoids of the same frequency, their derivatives, their integrals, and constants, the sum of the constant terms in both members of the equation must be equal (see Prob. 7) and so they may be cancelled out, which leaves an equation between sinusoids in which the integrals are not to be provided with integration constants.

§5. THE CORRESPONDENCE BETWEEN SINE FUNCTIONS AND COMPLEX NUMBERS.

Consider the totality of sine functions $A\sin(\omega t+\alpha)$ of the same frequency $\frac{|\omega|}{2\pi}(\neq 0)$ for all possible real values of the coefficient A and of the phase angle α. Let us assign to each sine function $A\sin(\omega t+\alpha) = A'\sin\omega t + A''\cos\omega t$ (where $A' = A\cos\alpha$ and $A'' = A\sin\alpha$) of this class a complex number $KA\underline{/\alpha} = K(A'+A''i)$, where K is any real number >0 (but the same for all the sine functions). To indicate this assignment (also called a mapping) we write:

$$A\sin(\omega t+\alpha) = A'\sin\omega t + A''\cos\omega t \longrightarrow KA\underline{/\alpha} = K(A'+A''i). \quad (1)$$

This mapping is single-valued, (once the constant K is chosen) because the coefficients A' and A'' of $\sin\omega t$ and $\cos\omega t$, respectively, are uniquely determined by $A\sin(\omega t+\alpha)$, and hence so is the corresponding complex number $K(A'+A''i)$; moreover, the correspondence is one to one, because distinct sine functions have at least one of the coefficients A' or A'' distinct, and hence the corresponding complex numbers are also distinct. To indicate that the correspondence established by the assignment (1) is one to one (or bi-unique), a double arrow is used, thus:

$$A\sin(\omega t+\alpha) = A'\sin\omega t + A''\cos\omega t \longleftrightarrow KA\underline{/\alpha} = K(A'+A''i). \quad (2)$$

By §4 we know that any (finite) linear combination:

$$\sum_k a_k\cdot A_k\sin(\omega t+\alpha_k) = \sum_k (a_kA_k)\sin(\omega t+\alpha_k),$$

of sine functions $A_k\sin(\omega t+\alpha_k)$ of the same frequency, with real coefficients a_k, is also a sine function $A\sin(\omega t+\alpha)$, say, of the same frequency, which may be obtained by eq. (1), §4 (p. 99). We shall now show that the one to one correspondence (2) preserves (finite) linear combinations, in the sense that the complex number $KA\underline{/\alpha}$ corresponding to the linear combination is equal to the corresponding linear combination $\sum a_k\cdot KA_k\underline{/\alpha_k} = K\cdot\sum a_kA_k\underline{/\alpha_k}$ of the complex numbers $KA_k\underline{/\alpha_k}$ corresponding to the sine functions which were combined; that is,

if
$$A_k\sin(\omega t+\alpha_k) \longleftrightarrow KA_k\underline{/\alpha_k}$$
and
$$\sum_k a_kA_k\sin(\omega t+\alpha_k) = A\sin(\omega t+\alpha) \longleftrightarrow KA\underline{/\alpha},$$
then
$$KA\underline{/\alpha} = \sum_k a_k\cdot KA_k\underline{/\alpha_k} = K\cdot\sum_k a_kA_k\underline{/\alpha_k};$$
or in short:
$$\sum_k a_k\cdot A_k\sin(\omega t+\alpha_k) \longleftrightarrow \sum_k a_k\cdot KA_k\underline{/\alpha_k} = K\sum_k a_kA_k\underline{/\alpha_k}. \quad (3)$$

This is easily proved as follows. Let: $A'_k = A_k\cos\alpha_k$ and $A''_k = A_k\sin\alpha_k$. Then: $A_k\sin(\omega t+\alpha_k) = A'_k\sin\omega t + A''_k\cos\omega t$, and we have:

$$\begin{aligned}\sum_k a_k\cdot A_k\sin(\omega t+\alpha_k) &= \sum_k a_k(A'_k\sin\omega t + A''_k\cos\omega t)\\ &= (\sum_k a_k A'_k)\sin\omega t + (\sum_k a_k A''_k)\cos\omega t\\ &\longleftrightarrow K(\sum_k a_k A'_k + i\sum_k a_k A''_k) = \sum_k a_k\cdot K(A'_k + A''_k i)\\ &= \sum_k a_k\cdot KA_k\angle\alpha_k ,\end{aligned}$$

which proves the statment.

Due to this property of the one to one correspondence between the class of complex numbers and the class of sine functions of a given frequency $\neq 0$, indicated by the relation (2), of preserving (finite) linear combinations, these two classes are said to be isomorphic with respect to (finite) linear combinations (with real coefficients); and the one to one correspondence is called an isomorphism; moreover, since the correspondence is one to one, equal sine functions shall be mapped into equal complex numbers and so equations between sine functions shall go over into equations between the corresponding complex numbers, i.e., this isomorphism shall also preserve equations.

In the following, it shall be convenient to denote a typical sine function, $A\sin(\omega t+\alpha)$, of a given 2π-frequency ω, by such a symbol as $s^*=s^*(t)$ and to denote the corresponding complex number simply by s. In this notation, if:

$$s^*_k = s^*_k(t) = A_k\sin(\omega t+\alpha_k) \longleftrightarrow s_k = KA_k\angle\alpha_k ,$$

the above result on the isomorphism between the classes of complex numbers and sine functions of a given frequency $\neq 0$ can be expressed thus:

$$\sum_k a_k s^*_k \longleftrightarrow \sum_k a_k s_k ; \qquad (4)$$

and if $s^* = s'^*$ then $s = s'$.

The isomorphism maps the null sinusoid: $0\sin(\omega t+\alpha)$, (the value of α does not matter here) into the complex number 0; it maps the sinusoid: $1^*(t)=K\sin\omega t$ into the number 1, and it maps the sinusoid: $i^*(t)=K\cos\omega t$ into the imaginary unit i; the sinusoid $\sin\omega t$ is mapped into the positive number K, and $\cos\omega t$ is mapped into Ki.

The isomorphism between the complex numbers and the sinusoids of a given frequency $\neq 0$ also has the following important properties:

If $s^*=s^*(t)=A\sin(\omega t+\alpha) \longleftrightarrow s=KA\angle\alpha$, then to the derivative, ds^*/dt, of $s^*(t)$, which is also a sinusoid of the same frequency, namely: $\omega A\cos(\omega t+\alpha) = \omega A\sin(\omega t+\alpha+90°)$ there corresponds the complex number $K\omega A\underline{/\alpha+90^\circ} = Ki\omega A\underline{/\alpha} = i\omega s$; and to the sinusoidal part of the integral $\int s^*(t)dt$ (without the integration constant), which is the sinusoid: $-\frac{A}{\omega}\cos(\omega t+\alpha) = \frac{A}{\omega}\sin(\omega t+\alpha-90^\circ)$, there corresponds the complex number $K\frac{A}{\omega}\underline{/\alpha-90^\circ} = K\frac{A}{i\omega}\underline{/\alpha} = s/i\omega$. In symbols we then have that if: $s^* = A\sin(\omega t+\alpha) \longleftrightarrow s = KA\underline{/\alpha}$, then:

$$\frac{ds^*}{dt} \longleftrightarrow i\omega s \qquad \text{and} \qquad \int_* s^* dt \longleftrightarrow \frac{s}{i\omega}, \tag{5}$$

where $\int_*$ denotes integration without the integration constant.

Thus we see that differentiation amongst the class of sine functions of angular frequency $\omega \neq 0$ corresponds to multiplication by ωi in the class of complex numbers, and integration (without adding the integration constant) corresponds to division by ωi.

Problem 1. Show that the product of two non-zero sine functions of the class of sinusoids of angular frequency $\omega \neq 0$ is not a sine fuction of this class. Consequently there is no hope in seeking a complex number representing this product by means of the isomorphism (2). Incidently, the same can be said about the integral of a sine function, with a non-zero integration constant.

Problem 2. Show that any given equation between linear combinations of sinusoids of the same angular frequency $\omega \neq 0$ (with real coefficients) can be written as an equation such as:

$$\sum_n A_n \sin(\omega t + \alpha_n) = 0. \tag{6}$$

By using the isomorphism (2) transform this equation into an equation amongst complex numbers and show that the Gauss-Argand diagram of the resulting equation yields a closed polygon, called the vector diagram of the given equation; in particular, any one of the sinusoids of the given equation can be obtained graphically from the vector diagram of the equation by finding the sine function corresponding to the resultant of all the other vectors corresponding to the other terms of the given equation, expressed in the form (6).

Problem 3. The value of $\underline{K}$ appearing in the mapping (2) can be chosen so as to make all the complex numbers dealt with in any particular discussion have moduli not greater (or not smaller) than a suitable size. This is sometimes important for numberical and graphical computations, or for vector diagrams. For example, to compute the sum:

$$s^*(t) = 391\sin(\omega t-10^\circ) - 505\sin(\omega t+127^\circ) + 276\sin(\omega t+83^\circ)$$

we can choose $K=1/100$; then, transforming all the sinusoids into complex numbers by means of the isomorphism: $A\sin(\omega t+\alpha) \longleftrightarrow \frac{A}{100}\underline{/\alpha}$,

we obtain: $s = 3.91\underline{/-10^\circ} - 5.05\underline{/127^\circ} + 2.76\underline{/83^\circ}$.
Find the sum of these complex numbers (graphically or otherwise) and then transform the result back into a sine function of angular frequency ω in order to obtain $s^*(t) = 100|s| \sin(\omega t + \text{Ang } s)$.
In a similar way, obtain the following sum (choosing $K = 100$, say):

$$s^*(t)=0.0537 \sin(400t+36^\circ)+0.000121\frac{d}{dt}\sin(400t-98^\circ)+194\int_* \sin(400t-8^\circ)dt.$$

Problem 4. Assuming that all the coefficients a_{mn}, b_{mn}, c_{mn} are real constant, and that all the functions $f_n^*(t)$ and $g_n^*(t)$ are sine functions of the same angular frequency $\omega \neq 0$, show that the system of integro-differential equations:

$$\sum_{n=1}^{N}(a_{mn}f_n^*(t) + b_{mn}\frac{df_n^*}{dt} + c_{mn}\int f_n^*(t)dt) = g_m^*(t) \quad (m=1,2,\ldots,N)$$

is transformed into the algebraic system:

$$\sum_{n=1}^{N}(a_{mn} + i\omega b_{mn} + c_{mn}/i\omega)\, f_n = g_m \qquad (m=1,2,\ldots,N)$$

by means of the isomorphism (2).

Note. In the literature the isomorphism mostly used is the following (although this is not always explicitly stated):

$$A \sin(\omega t + \alpha) \longleftrightarrow \frac{A}{\sqrt{2}}\underline{/\alpha} \; ; \tag{7}$$

in other words, the value $K = 1/\sqrt{2}$ is taken. This has many advantages, because the modulus of the complex number corresponding to any sinusoid is then equal to its effective, or root-mean-square (rms), value, and most instruments in use for the measurement of currents and voltages are calibrated so as to read effective, or rms, values. Thus the reading of such an instrument shall give the modulus of the complex number when the quantity measured is sinusoidal. In the following, when no explicit mention is made of the isomorphism between complex numbers and sinusoids of the same frequency to be used, we shall always understand that the particular isomorphism (7) is used.

Suppose now that in $z=r\underline{/\vartheta}=re^{\vartheta i}$, $\underline{r}$ and $\underline{\vartheta}$ are functions of the time $\underline{t}$. Then $\underline{z=z(t)}$ shall represent (in the Gauss-Argand diagram) a moving vector, and its tip, or terminal, a moving point. When $\vartheta=\vartheta(t)$ is a steadily increasing function of the time, $\underline{z(t)}$ is called a rotating vector, although this name is usually understood to mean that $\underline{r}$=constant and that $\vartheta=\omega t+\alpha$, with ω and α constant (unless otherwise stated). In this latter case, (the tip of) $\underline{z(t)}$ shall represent a point moving on a circular path at a constant angular velocity ω, and α will denote the initial angle (with the x-axis).

Problem 5. Show that the following mapping (for fixed $\omega \neq 0$ and for fixed $K \neq 0$):

$$A e^{i(\omega t+\alpha)} = A e^{i\alpha} e^{i\omega t} \longrightarrow K A \underline{/\alpha}\,, \tag{8}$$

is an isomorphism with respect to linear combinations and equations (i.e. which preserves linear combinations and equations) which carries differentiation over into multiplication by $i\omega$ and integration (without the additional integration constant) into division by $i\omega$.

Problem 6. Show that the following mapping (for fixed $\omega \neq 0$):

$$A e^{i(\omega t+\alpha)} \longrightarrow A \sin(\omega t+\alpha) \tag{9}$$

is also an isomorphism with properties similar to those of (8).

Note. From the results of problems 5 and 6 concerning the isomorphisms (8) and (9), one can obtain the properties of the fundamental isomorphism (2), between the complex numbers and the sinusoids of a given frequency $\neq 0$, mentioned above in the text. In fact, in the older literature, this was the way the thing was done, although the isomorphism (9) was rarely mentioned explicitly. Its use was hidden by actually starting out with the so called rotating vectors (instead of with the actual physical sinusoids) and then using a technique equivalent to the use of the mapping (8). In this way the concept of a rotating vector was practically necessary from the start and this was a cause of many a student failing to understand alternating currents. By using the isomorphism (2) one can go directly from the physical sinusoids to the auxiliary, but helpful, complex numbers without passing through the rotating vectors and then, once an analysis is made in terms of the auxiliary complex numbers, one may return with the results back to the physical sinusoids via the same isomorphism (2). The rotating vectors then become unnecessary, in this connection, and mention of them is only made for reference.

Problem 7. Suppose that we have a set of sine functions of the time t, all of the same angular frequency $\omega \neq 0$. Suppose that we make a Gauss-Argand diagram of all the complex numbers obtained by the isomorphism (2) for all the sine functions of the given set. Suppose that we now rotate this diagram as a rigid structure with an angular velocity ω around the origin. Show that we get the diagram of the rotating vectors which we would obtain for the given set of sine functions according to the isomorphism (9). In this way it can be seen that any configuration of the set of rotating vectors can be obtained by a suitable rigid rotation of the configuration of "stationary" or fixed vectors obtained by the mapping (2).

Note. According to the preceding problem (7), the mutual relations between the rotating vectors corresponding to a given set of sine functions of the same angular frequency do not change with the time, and since by a proper change of the time origin we can send the initial line (the real axis) into any position, according to Prob. 1, §4 (p. 98), we see that the coordinate axes in a vector diagram of a set of sine functions are irrelevant; it should be understood that the axes can be supplied in any arbitrary position, in the sense that it is always possible to choose an origin of the time for which the resulting diagram represents the (fixed or rotating) vectors corresponding to the various sine functions of the given set. This is the reason why vector diagrams corresponding to sine functions of the same angular frequency are frequently given without coordinate axes.

§6. EXPONENTIALLY MODULATED SINUSOIDS.

The results of the preceding section (5) can easily be extended to the so-called damped harmonic oscillations (or damped sinusoids) of the form: $A\,e^{\sigma t}\sin(\omega t+\alpha)$, where $\sigma<0$ and $\omega\neq 0$; or more general, to the exponentially modulated harmonic oscillations (exponentially modulated sinusoids) of this same form, but without the restriction $\sigma<0$.

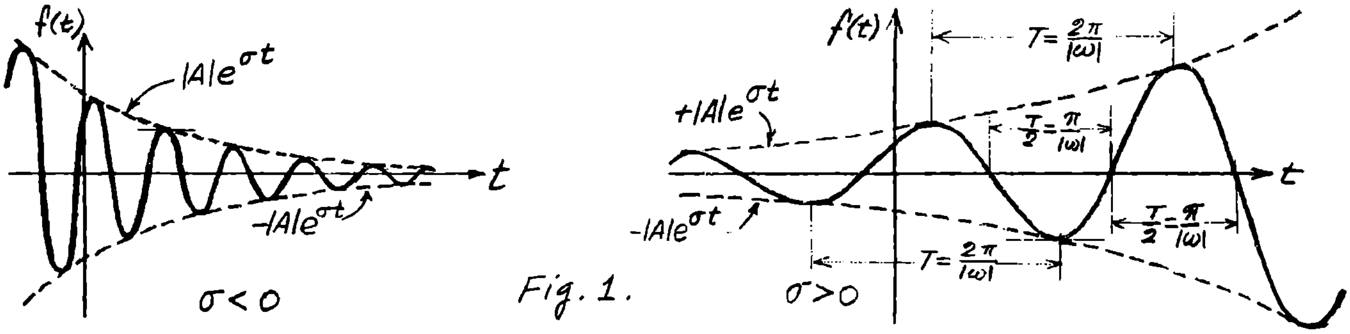


Fig. 1.

The graphs of typical exponentially modulated sinusoids are shown above (Fig. 1) for the two cases $\sigma<0$ and $\sigma>0$ (the case $\sigma=0$, corresponding to an ordinary sinusoid, which was treated amply before, being omitted).

The function given by: $f(t) = A\,e^{\sigma t}\sin(\omega t+\alpha) = A'e^{\sigma t}\sin\omega t + A''e^{\sigma t}\cos\omega t$ vanishes for all values of t for which the sine factor $\sin(\omega t+\alpha)$ vanishes (and only for these, if $A\neq 0$); that is, the zeros of $f(t)$ are given by the solutions of the equation: $\omega t+\alpha = n\pi$, where n is any integer, and therefore, they are distributed uniformly along the t-axis at intervals of size: $\pi/|\omega|$. ($A' = A\cos\alpha$ & $A'' = A\sin\alpha$, above.)

Since $|\sin(\omega t+\alpha)| \leqq 1$ for all t, and since $\sin(\omega t+\alpha) = \pm 1$ for infinitely many values of t, we see that the graph of $f(t)$ is contained between the enveloping (limiting) curves: $y=\pm|A|e^{\sigma t}$ (shown by the dotted lines in Fig. 1), touching them alternatively at equal intervals of the same size: $T/2 = \pi/|\omega|$. The points of contact between $f(t)$ and the limiting curves are given by the solutions of the equation: $\omega t+\alpha = (4n\pm1)\cdot\pi/2$, where n is any integer (the upper sign corresponds to contacts with the upper limiting curve and the lower sign to contacts with the lower curve).

The values of t for which $f(t)$ attains a maximum or a minimum can be found by equating its first derivative to zero. In this way we get the equation: $\tan(\omega t+\alpha) = -\omega/\sigma$, satisfied by the points of maxima and minima, of which the solutions are: $t_n = \frac{1}{\omega}(n\pi-\alpha-\tan^{-1}\frac{\omega}{\sigma})$, for all integers n. These values of t do not coincide with the corresponding points of contact with the limiting curves, unless $\sigma = 0$. The maxima and minima occur alternatingly, and the (time) interval, (say), $t_{n+2} - t_n$, between two successive maxima (or minima) is a constant, namely: $2\pi/|\omega|$, which is precisely the period, T (say), of the sine factor in $f(t)$. The maxima and minima of $f(t)$ are: $f(t_n)=Ae^{\sigma t_n}\sin(\omega t_n+\alpha)$. From this we see that the maxima (and the minima) vary exponentially, and the ratio of two successive maxima (or minima) is a constant:

$$f(t_n)/f(t_{n+2}) = A\,e^{\sigma t_n}\sin(\omega t_n+\alpha)/A\,e^{\sigma t_{n+2}}\sin(\omega t_{n+2}+\alpha) = e^{-2\pi\sigma/|\omega|},$$

since $t_{n+2} = t_n + 2\pi/|\omega| = t_n + T$. The natural logarithm of this constant ratio, namely: $-\frac{2\pi\sigma}{|\omega|}$, is known as the logaritmic decrement, δ (say), of $f(t)$, speaking algebraically (with a negative logarithmic decrement meaning a logarithmic increment).

Concerning the limiting curves of $f(t)$, the exponential factor in $f(t)$, namely: $e^{\sigma t}$, is called a damping factor (speaking algebraically) and the coefficient of t in the exponent, namely σ, is called the damping constant or coefficient (algebraically speaking; so that a negative damping coeff. would mean an amplifying coeff.). The constant (time) interval in which the damping factor diminishes by the factor $1/e$ is called the (time)-constant, τ (say), of $f(t)$; thus $A\,e^{\sigma(t+\tau)}/A\,e^{\sigma t}=1/e$, so that: $\sigma\tau = -1$.

Hence we have the following relations between the logarithmic decrement δ, the time constant τ, and the period T and the angular frequency $|\omega| = 2\pi/T$ of the sine factor, of an exponentially modulated sine function: $\tau\delta = T = 2\pi/|\omega| = -\delta/\sigma$.

Problem 1. If $f(t)$ and $g(t)$ are two exponentially modulated sinusoids of the same damping coefficient and angular frequency (or of the same time constant and logarithmic decrement), show that $a\,f(t)$ (a=real) and $f+g$ are also exponentially modulated sinusoids with the same damping coefficient and angular frequency. Thence show that any finite linear combination (with real coefficients) of such functions is also one such function.

Problem 2. Show that the derivative and the indefinite integral of an exponentially modulated sinusoid are also functions of this type with the same time constant and logarithmic decrement.

Let us now consider the totality of (real) exponentially modulated sinusoids of the real variable t (time, say) with the same damping coefficient and angular frequency; that is, the class of functions of the form: $A\,e^{\sigma t}\sin(\omega t+\alpha)$ with the same σ and ω ($\neq 0$), for all possible real values of A and α. To each function of this class, let us assign a complex number as indicated by the mapping:

$$A\,e^{\sigma t}\sin(\omega t+\alpha) \longrightarrow KA\underline{/\alpha}\,, \tag{1}$$

where K is any (fixed) number > 0, the same for all the functions.

From the results obtained above (in §5) for the isomorphism between the class of sine functions of the same frequency and the class of complex numbers, we can easily see that the mapping (1) establishes a one to one correspondence between the complex numbers and the exponentially modulated sinusoids of the same damping coefficient and angular frequency, which preserves (finite) linear combinations with real coefficients, and equations between such; so that the mapping (1) is indeed also an isomorphism with respect to these relations.

Therefore, denoting an exponentially modulated sinusoid by a starred letter and the corresponding complex number by the letter used, we shall have (where a_n and b_m are any real numbers):

$$\sum_n a_n f_n^*(t) = \sum_m b_m g_m^*(t), \text{ if and only if: } \sum_n a_n f_n = \sum_m b_m g_m. \tag{2}$$

Moreover, for the derivative of an exponentially modulated sine function we have (where: $f^* = A\,e^{\sigma t}\sin(\omega t+\alpha)$):

$$\frac{df^*}{dt} = A\omega e^{\sigma t}\cos(\omega t+\alpha) + A\sigma e^{\sigma t}\sin(\omega t+\alpha) \longleftrightarrow KA\omega\underline{/\alpha+90^\circ} + KA\sigma\underline{/\alpha} = (\omega i+\sigma)KA\underline{/\alpha} = (\sigma+\omega i)f;$$

and for the indefinite integral (omitting the integration constant):

$$\int f^* dt = \frac{A e^{\sigma t}}{\sigma^2+\omega^2}\left[\sigma\sin(\omega t+\alpha) - \omega\cos(\omega t+\alpha)\right] \longleftrightarrow \frac{AK}{\sigma^2+\omega^2}\left(\sigma\underline{/\alpha} - \omega\underline{/\alpha+90^\circ}\right) = \frac{AK}{\sigma^2+\omega^2}(\sigma-\omega i)\underline{/\alpha} = \frac{KA\underline{/\alpha}}{\sigma+\omega i} = \frac{f}{\sigma+\omega i}.$$

Hence, putting $p=\sigma+\omega i$, we have that if: $f^*(t) \longleftrightarrow f$, then:

$$\frac{df^*(t)}{dt} \longleftrightarrow p\cdot f \qquad \text{and} \qquad \int_* f^*(t)dt \longleftrightarrow f/p; \tag{3}$$

so that differentiation is carried over into multiplication by $p=\sigma+\omega i$, and indefinite integration is carried over into division by p. Thus, the mapping (1) transforms differentiation and integration (without adding the integration constant) amongst the exponentially modulated sine functions into multiplication and division, respectively, of the corresponding complex numbers, by the fixed complex number $p=\sigma+\omega i$.

Problem 3. Express the following sum:

$$f^*(t)=57e^{3.6t}\sin(900t+60^\circ)-\frac{d}{dt}0.15e^{3.6t}\sin(900t-20^\circ)+66000\int_* e^{3.6t}\sin(900t-120^\circ)dt,$$

where $\int_*$ denotes the indefinite integral (without the integration constant), as an exponentially modulated sinusoid. (In (1), take: $K=10^{-1}$.). Interpret the equation as a closed polygon in the complex plane.

Problem 4. Assuming that all the coefficients a_{mn}, b_{mn}, c_{mn}, are real constants, and that all the functions $f_n^*(t)$ and $g_n^*(t)$ are exponentially modulated sinusoids of the same time constant and logarithmic decrement ($\neq 0$), show that the following system of integro-differential equations: (using Prob. 5 to get rid of the integ. const.)

$$\sum_{n=1}^{N}\left[a_{mn}f_n^*(t) + b_{mn}\frac{d}{dt}f_n^*(t) + c_{mn}\int f_n^*(t)dt\right] = g_m^*(t), \qquad (m=1,2,\ldots,N) \quad (4)$$

is transformed into the following algebraic system: (where $p=\sigma+\omega i$)

$$\sum_{n=1}^{N}(a_{mn} + b_{mn}p + c_{mn}/p)\, f_n = g_m, \qquad (m=1,2,\ldots,N)$$

by means of the isomorphism (1), which carries f_n^* and g_n^* into f_n and g_n.

Problem 5. Show that if (where $A'=A\cos\alpha$ and $A''=A\sin\alpha$):

$$A e^{\sigma t}\sin(\omega t+\alpha) + B = A'e^{\sigma t}\sin\omega t + A''e^{\sigma t}\cos\omega t + B = 0$$

for all t, where A, α, and B, are constant, then: $A = A' = A'' = B = 0$.

It is interesting to compare the results of this section (6) with the results of the preceding section (5). The isomorphism (1) of §5, between the sine functions and the complex numbers, and the isomorphism (1) (of this section), between the exponentially modulated sinusoids and the complex numbers, both preserve (finite) linear combinations and equations, but differentiation and integration (without the additional integration constants) are carried by the former into multiplication and division by ωi (respectively) while they are carried by the latter into multiplication and division by $p=\sigma+\omega i$ (respectively). In particular, for linear integro-differential equations with constant coefficients, we see that in the case of exponentially modulated sinusoids of the form $A e^{\sigma t}\sin(\omega t+\alpha)$, $p=\sigma+\omega i$ takes the place of the ωi for the case of sinusoids of the form $A\sin(\omega t+\alpha)$. Thus we see that the purely imaginary complex number ωi, which corresponds to the angular frequency ω in the case of sine functions, is generalized to an arbitrary complex number $p=\sigma+\omega i$ in the case of exponentially modulated sinusoids. For this reason the complex number $p=\sigma+\omega i$ is frequently called a generalized or (complex) frequency. This concept of a generalized (complex) frequency in the complex plane is very useful in the interpretation of results concerning linear integro-differential equations with real constant coefficients, such as those of constant parameter electric networks, with sinusoidal or exponentially modulated sinusoidal exciting functions (the known functions in the equations).

Example. Consider the following system of differential equations with real constant coefficients a_{kmn}:

$$\sum_{m=1}^{M}\sum_{n=0}^{N} a_{kmn}\frac{d^n}{dt^n}f_m^*(t) = g_k^*(t), \qquad (k=1,2,\ldots,M) \qquad (5)$$

where the known (exciting) functions $g_k^*(t)$ are exponentially modulated sinusoids of the same damping coefficients σ and angular frequency ω, and let us seek functions of the same kind for the unknown (response) functions $f_m^*(t)$ satisfying the differential equations.

By means of the isomorphism (1), the above differential equations are transformed into the following algebraic equations:

$$\sum_{m=1}^{M}\sum_{n=0}^{N}(a_{kmn}p^n)f_m = \sum_{m=1}^{M} b_{km}f_m = g_k, \qquad (k=1,2,\ldots,M)$$

where f_k and g_k $(k=1,2,\ldots,M)$ are the complex numbers corresponding to $f_k^*(t)$ and $g_k^*(t)$, and $b_{km}=\sum_{n=0}^{N}a_{kmn}p^n$ are known complex numbers dependent on $p=\sigma+\omega i$. If the determinant of the b_{kl} is not zero, the solution of these equations is given by:

$$f_m = \sum_{k=1}^{M} g_k \operatorname{cof} b_{km}\Big/\det(b_{kl}), \qquad (m=1,2,\ldots,M)$$

and then we can find the unknown (time) functions, $f_m^*(t)$, as follows:

$$f_m^*(t) = K^{-1}|f_m|e^{\sigma t}\sin(\omega t + \operatorname{Ang} f_m). \qquad (m=1,2,\ldots,M)$$

Since each b_{kl} is a polynomial in p, so shall each of the cofactors and the determinant of the b_{kl} be, and hence we see that each f_m is a rational function of p. Consequently, there shall then be only a finite number of (uniquely determined) values at which any one of the f_m becomes infinite (called the poles of f_m)① and only a finite number of (uniquely determined) values of p at which f_m becomes zero (called the zeros of f_m), and the same for $|f_m|$.

When all the exciting functions have damping coefficients σ and angular frequencies ω such that the number $\sigma+\omega i$ is very near a zero of f_m, $|f_m|$ and hence the function $f_m^*(t)$ shall always be small; and if $\sigma+\omega i$ is very near a pole of f_m, then $|f_m|$ shall be very large and so $f_m^*(t)$ shall be exceptionally large during large intervals of t.

① The totality of the poles of all the response quantities f_m are called the generalized natural frequencies of the system (5).

§7. THE LINEAR INDEPENDENCE OF EXPONENTIALS (AND OF SINUSOIDS).

By a linear combination of the (arbitrary) functions f_k (k=1,2,.. with respect to (or over) an arbitrary class K of coefficients we mean any finite sum $\sum_k c_k f_k$ (which we assume to be defined) where all the "coefficients" c_k are elements of the class K. (For example, K may be the class of rational numbers, or the class of real numbers, or the class of polynomials in several given variables with complex coefficients, etc.)

The functions $f_1, f_2, \ldots, f_N$ are said to be linearly dependent over K if and only if there exists a linear combination $\sum_{n=1}^{N} c_n f_n$ of these functions over K which is identically zero (i.e. zero for all values of the variables in the f_n) with not all the coefficients c_n $(n=1,2,\ldots,N)$ equal to zero. (To be identically zero is sometimes indicated by writing $\equiv 0$.)

The linear independence of $f_1, f_2, \ldots, f_N$ over K means that they are not linear dependent over K. These functions shall be linearly independent over K if and only if $\sum_{n=1}^{N} c_n f_n = 0$ implies that all the coefficients c_n $(n=1,2,\ldots,N)$ are zero. For example, it is a well known result of algebra that any polynomial (in any number of variables) with complex numbers as coefficients which vanishes identically must have all its coefficients equal to zero. Consequently, the products of distinct non-negative powers of any number of variables are linearly independent over the class of complex numbers.

Problem 1. Show that if $f=0$ then the functions $f, f_1, f_2, \ldots, f_N$ are linearly dependent over every class of "coefficients".

We shall now prove the following important theorem:

Any finite set of exponentials $e^{a_n t}$ $(n=1,2,\ldots,N)$, where the a_n are distinct complex numbers, are linearly independent over the class of polynomials in the variable t with complex numbers as coefficients.

In other words, if we have:

$$\sum_{n=1}^{N} P_n(t)\, e^{a_n t} = 0 \tag{1}$$

where all the a_n are distinct complex numbers and all the $P_n(t)$ are polynomials in t with complex numbers as coefficients, then we must have all $P_n(t) = 0$ $(n=1,2,\ldots,N)$; and this in turn means that all the coefficients of all the polynomials $P_n(t)$ vanish.

Proof: This we do by mathematical induction on the number N.

For $N=1$ this is trivial; because if $P(t)\,e^{at}=0$ then $P(t)=0$, since $e^{at} \neq 0$ for all t.

Now assume the theorem true for all $n \leqq N$. We wish to infer its truth for $n=N+1$. To do this, let us suppose that we have:

$$\sum_{n=1}^{N+1} P_n(t)\, e^{a_n t} = 0, \qquad (2)$$

where some one of the polynomial coefficients is not zero, say: $P_1(t)$. Let b be the leading coefficient in $P_1(t)$, i.e. the (non-vanishing) coefficient of the highest power of t in $P_1(t)$. Since $\exp(a_{N+1} t)$ is never zero, we can divide by it and we get:

$$\sum_{n=1}^{N} P_n(t) \exp{(a_n - a_{N+1})t} + P_{N+1}(t) = 0.$$

If we now differentiate this a sufficient number of times (equal to the degree δ of P_{N+1}, plus 1) we obtain an equation of the form:

$$\sum_{n=1}^{N} Q_n(t) \exp{(a_n - a_{N+1})t} = 0,$$

where all the $(a_n - a_{N+1})$, for $n=1,2,\ldots,N$, are distinct (since all the a_n are distinct) and where the $Q_n(t)$ are also polynomials in t (in fact, of the same degrees as the corresponding $P_n(t)$). Moreover, the leading coefficients in the $Q_n(t)$ are equal to $(a_n - a_{N+1})^{\delta+1}$ multiplied by the corresponding coefficients in the $P_n(t)$. But by the induction hypothesis all the $Q_n(t)$ vanish (and in particular, $Q_1(t)=0$); hence the leading coefficient of $Q_1(t)$, which is $(a_1 - a_{N+1})\, b$, must be zero; so that $b = 0$, since $a_1 - a_{N+1} \neq 0$. Consequently, $P_1(t)$ can have no leading coefficient and this means that $P_1(t)$ vanishes identically. Therefore the expression in eq. (2) reduces to an expression with N terms to which the induction hypothesis again applies, and thus we infer that all the other $P_n(t)$, for $n=2,3,\ldots,N+1$, also vanish identically. In this way the truth of the theorem for $n=N+1$ is proved under the hypothesis of its truth for $n \leqq N$; and since it is true for $n=1$, its truth for all positive integers n follows.

Calling any finite sum of products of polynomials by exponential a pe-function, the theorem given above can be stated briefly thus: a pe-function with distinct exponentials vanishes identically if and only if all the coefficients of all the polynomials vanish.

Corollary 1. A pe-function with a non-zero coefficient which vanishes identically must necessarily have two of its exponentials equal (i.e. at least two of the a_n in eq. (1) are necessarily equal).

Corollary 2. In an equation between pe-functions corresponding coefficients are equal. For if we have an equation of the form:

$$\sum_{n=1}^{N} P_n(t)\, e^{a_n t} = \sum_{n=1}^{N} Q_n(t)\, e^{a_n t},$$

then we will have: $\sum_n (P_n - Q_n)\, e^{a_n t} = 0,$

and so when all the a_n $(n=1,2,\ldots,N)$ are distinct we must have $P_n=Q_n$ $(n=1,2,\ldots,N)$ for all t, and this means precisely that corresponding coefficients are equal. The uniquely determined coefficients, a_n, of t in the exponentials of a pe-function are called the generalized (angular) frequencies of the pe-function.

Thus a single equation between pe-functions can be split into as many equations between polynomials as there are distinct exponentials; and, conversely, any number of equations between polynomials can be combined into a single equation between pe-functions, simply by multiplying the equations by distinct exponentials and adding. (Of course, each identity between polynomials can in turn also be split into several equations between corresponding coefficients.)

Consider now any finite sum of products of polynomials by sine functions, of the following form:

$$P(t) + \sum_{n=1}^{N} \left[P_n(t) \sin \omega_n t + Q_n(t) \cos \omega_n t\right], \tag{3}$$

where in general the ω_n are any complex numbers $\neq 0$ and $P(t)$ and the $P_n(t)$ and $Q_n(t)$ $(n=1,2,\ldots,N)$ are polynomials in t.

By means of Euler's formulas (cf. §2, Prob. 4, p. 87):

$$\sin \omega_n t = (e^{i\omega_n t} - e^{-i\omega_n t})/2i, \qquad \cos \omega_n t = (e^{i\omega_n t} + e^{-i\omega_n t})/2,$$

the expression (3) can be put into the following form:

$$P(t) + \frac{1}{2}\sum_{n=1}^{N}\left[(Q_n - P_n i)\, e^{i\omega_n t} + (Q_n + P_n i)\, e^{-i\omega_n t}\right]. \tag{4}$$

This is a pe-function because all the factors $(Q_n \pm P_n i)/2$ are also polynomials in t with complex numbers as coefficients. Hence, if all the ω_n are distinct and the expressions (3) and (4) vanish identically, we infer that:

$$P(t) = 0, \qquad (Q_n - P_n i)/2 = 0, \quad \text{and} \quad (Q_n + P_n i)/2 = 0,$$

for all t and all $n=1,2,\ldots,N$; and so, by adding and subtracting, we obtain: $Q_n(t) = 0$ and $P_n(t) = 0$, for all t and all $n=1,2,\ldots,N$, which means that all the coefficients of all the polynomials coefficients $P_n(t)$, $Q_n(t)$, and $P(t)$, vanish.

In this way we see that the theorem and corollaries given above for pe-functions hold also for expressions of the form (3).

The theorems of this section have certain importance for systems of linear integro-differential equations with constant coefficients and pe-exciting functions; because it can be shown that the solutions of such systems are then also pe-functions, and by the theorems given above each equation of the system can be split into various algebraic equations between the coefficients, which may be solved, and with the results the solutions of the systems of integro-differential equations can then be reconstructed.

CHAPTER V: ARBITRARY NETWORKS IN THE SINUSOIDAL STATE.

In Ch. III the general integro-differential equations of an arbitrary network were established. When these equations are linear and consistent, the general solution consists of a part which is a solution of the reduced equations in which all the exciting functions are replaced by zeros, called the complementary functions (but which do not necessarily form a solution of the original equations) and which contains all the arbitrary constants, and a part, containing no arbitrary constants, which is a particular solution.

In (stationary) alternating current networks, in which all the parameters are (assumed to be) constant, it is a well known result from the theory of linear differential equations with constant coefficients that the complementary functions always form a class of pe-functions (i.e. finite sums of products of polynomials by exponentials cf. Ch. IV, §7), whose exponentials are uniquely determined (except for their order, of course) in any given case, and which do not depend on the exciting (the known) functions. The uniquely determined set of the coefficients of the independent variable (the time, say) in the exponents of the exponentials are called the generalized (complex) natural frequencies of the network (see the footnote to the example of Ch. IV, §6, p.110) and of the class of pe-functions also. The particular solution, on the other hand, depends on the exciting functions, and on the coefficients in the system of integro-differential equations of the network which, in turn, depend only on the parameters of the network and on its combinatorial structure.

The fundamental alternating current problem is that of an arbitrary (stationary) network with constant parameters and sinusoidal exciting functions of the same angular frequency $\omega \neq 0$ which is not a natural angular frequency (meaning that $0+\omega i$ is not a generalized natural frequency) of the network. In the particular case of such a network, the particular solution of the corresponding system of equations consists also of sine functions of the same angular frequency ω, and the complete solution, equal to the complementary functions plus the particular solutions, consists of a set of pe-functions, the generalized frequencies of which are the generalized natural frequencies of the network and ωi. According to §7 of Ch. IV, due to the linear independence of distinct exponentials with polynomial coefficients, the system of equations of the network can then be split into various independent systems of equations, one system for each generalized natural frequency of the network, and a system corresponding to the angular frequency ω. The solutions of the systems of equations corresponding to the generalized natural frequencies make up the complementary functions and, when all the generalized natural frequencies of the network have negative real parts, the pe-functions of the complementary functions die out rapidly. One is then left only with the system of equations corresponding to the angular frequency ω of the applied sources which is of any practical significance, and in which all the (known and unknown) functions are sinusoids of the same angular frequency ω; in this way one obtains precisely the system of equations of the network in the sinusoidal state, the solution of which is a particular solution of the original system of equations of the network. When not all the generalized natural frequencies of the network have negative real parts, the pe-functions of the complementary functions do not in general die out and the currents and voltages in the elements of the network may differ considerably from the sine functions of the particular solution. Fortunately the cases in which the difference is considerable are rare in practice, and generally the sinusoids of angular frequency ω of the particular solution will represent the response quantities (currents and voltages) almost exactly (as time elapses). Nevertheless, the problem of finding the sinusoidal solution of the system of equations of a network, of the same frequency as that of all the sources, is (if it exists) in any case important in itself; and it may well be called the restricted a-c problem of the network, and most of what follows will be dedicated to it; but it should be understood that, in case not all the generalized natural frequencies of the network have negative real parts, there may also be other parts of the solution of practical significance (not being negligible), besides the sinusoidal parts, which must be determined by a separate analysis.

§1. NETWORKS OF GENERAL SERIES ELEMENTS IN THE SINUSOIDAL STATE.

Consider a (stationary) network of $\underline{n_e}$ general series elements (of the kind shown in Fig. 1), arbitrarily oriented and numbered consecutively from $\underline{1}$ to $\underline{n_e}$, and connected into $\underline{n_c}$ components (=separate parts) with $\underline{n_n}$ nodes. Omitting exactly one node in each component, let the other nodes be arbitrarily numbered consecutively from $\underline{1}$ to $\underline{n'_n}=n_n-n_c$.

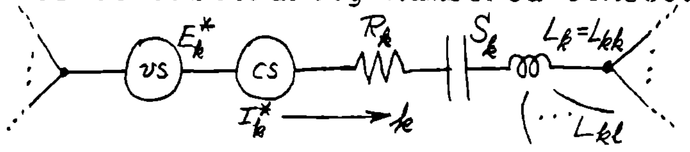


Fig. 1.

The components can be numbered $1, 2, \ldots, n_c$, and the node omitted in the component $a (= 1, 2, \ldots, n_c)$ will be taken as the base or reference node in that component, denoted 0_a. Let a complete and independent set of $n_m = n_e - n_n' = n_e + n_c - n_n$ meshes be chosen in the network and arbitrarily oriented and numbered consecutively from 1 to n_m. (Cf. end of Ch. II, §4, p. 60.)

Assume all the currents and voltages in the (elements of the) network to be sine functions of the same angular frequency $\omega \neq 0$. Since the complex numbers assigned to the sine functions according to the mapping (1), or (2), of Ch. IV, §5 (p. 101), will be used much more than the corresponding sinusoids, it shall be convenient (as was done in Ch. IV, §5) to denote the sine functions by starred letters (i.e. letters with an asterisk, such as E^*, V^*, I^*, etc.), and to denote the corresponding complex numbers simply by the letters used (i.e. the letters without the asterisks, namely, E, V, I, etc.). Then the equations of the network in the sinusoidal state shall be:

$$V_k^* = R_k I_k^* + S_k \int I_k^* \, dt + \sum_{l=1}^{n_e} L_{kl} \, dI_l^*/dt - E_k^* - D_k^*, \quad (k = 1, 2, \ldots, n_e)$$

$$\sum_{k=1}^{n_e} (k,n) \, I_k^* = 0, \quad (n = 1, 2, \ldots, n_n') \qquad \sum_{k=1}^{n_e} [k,m] \, V_k^* = 0, \quad (m = 1, 2, \ldots, n_m) \tag{1}$$

where V_k^* is the sinusoidal voltage drop in the general serie element k, I_k^* is the sinusoidal current through the element k, and E_k^* and D_k^* are the sinusoidal voltage rises in the voltage source and current source, respectively, in the element k, in the assigned reference direction. (These are the same equations one would obtain from the general equations (3), (4) & (5) of Ch. III, §1 (pp. 63-64) in the case that all the exciting functions are sinusoids of angular frequency ω, assumed not to be a natural angular frequency of the network, by equating the sinusoidal terms of angular frequency ω in each equation, based on their linear independence and knowing that all the functions of t in these equations are pe-functions; whether the other terms of the pe-functions die out, or not, as time elapses.)

Let us now transform each equation of this system (1) into an equation between complex numbers, according to the isomorphism:

$$s^* = s^*(t) = A \sin(\omega t + \alpha) \longleftrightarrow K A \underline{/\alpha} = s \tag{2}$$

(cf. Ch. IV, §5, (1) or (2), p. 101) between the sinusoids of angular frequency ω and the class of complex numbers, where K is any real number > 0, usually taken $= 1/\sqrt{2}$ in order to have $|s| = \overline{\overline{s^*}}$ (i.e. so that the modulus, or absolute value, of the complex number be equal to the effective value of the corresponding sinusoid). [Unless mentioned explicitly otherwise, we shall always understand $K = 1/\sqrt{2}$ to be taken in the correspondence (2).]

We know that this isomorphism preserves linear combinations and equations and that differentiation is carried over into multiplication by ωi while integration (without adding the usual integration constants, which can be omitted due to their linear independence, or according to Prob. 7 of Ch. IV, §4, p. 99) is carried over into division by ωi. Therefore, the system of equations (1) is transformed into the following system:

$$\begin{gathered} V_k = R_k I_k + S_k I_k/\omega i + \sum_{l=1}^{n_e}\left(L_{kl}\cdot\omega i\, I_l\right) - E_k - D_k, \quad (k=1,2,\ldots,n_e) \\ \sum_{k=1}^{n_e}(k,n)\, I_k = 0, \quad (n=1,2,\ldots,n_n') \qquad \sum_{k=1}^{n_e}[k,m]\, V_k = 0, \quad (m=1,2,\ldots,n_m) \end{gathered} \tag{3}$$

assuming, of course, that all the network parameters, namely, all the resistances R_k, all the elastances S_k, and all the (self- and mutual-) inductances L_{kl}, are constant.

These equations (3) are called the transformed equations of the network of general series elements in the sinusoidal state. They are also known in the literature as the complex, vectorial, or symbolic, equations of an alternating current network (in the sinusoidal state). The complex numbers V_k and I_k corresponding to the sinusoidal voltage drops V_k^* and currents I_k^* through the elements $k(=1,2,\ldots,n_e)$ are called the transformed, or complex, voltage drops and currents, respectively, in the corresponding elements, in the assigned reference directions. The complex numbers E_k and D_k corresponding to the sinusoidal voltage rises E_k^* and D_k^* in the voltage and current sources, respectively, of the elements k are called the transformed, or complex, voltage rises in the corresponding sources. (In particular, E_k is called the complex electromotive force of the voltage source in the element k.) The complex number corresponding to the sinusoidal value of a (current or voltage) source will be called the complex value of the source. The (known) complex values of the sources of a network are called the complex exciting (or driving) quantities of the network (and of the corresponding transformed equations). In the literature, complex currents and voltages are also known as vector currents and voltages, respectively.

It can be noticed that if V_{R_k}, V_{S_k}, and V_{L_k}, are the complex voltage drops in the resistance R_k, in the elastance S_k, and in the inductance L_k, respectively, in the reference direction of the complex current I_k through them, then we shall have:

$$V_{R_k} = R_k I_k, \qquad V_{S_k} = \frac{S_k}{\omega i} I_k, \qquad V_{L_k} = \sum_{l=1}^{n_e} i\omega L_{kl} I_l, \quad (k=1,2,\ldots,n_e) \tag{4}$$

as can be observed from the separate terms of the first of the eqs. (3).

When the inductance L_k is magnetically isolated, the last of the eqs. (4) reduces, of course, to the equation: $V_{L_k} = i\omega L_k I_k$; and in any case this quantity is called the *self-complex voltage drop* in the inductance L_k, while the other terms composing V_{L_k}, namely, $i\omega L_{kl} I_l$ ($l=1,2,\dots,n_e$) are called the *mutual-complex voltage drops* in the coil L_k due to the complex currents I_l in the other coils *l*. The (total) complex voltage drop in the inductance L_k is then the sum of the self-and all the mutual-complex voltage drops in L_k.

The coefficients of the complex currents in the eqs. (4) are called the *impedances* (a term due to Oliver Heaviside) of the basic elements; specifically, $\underline{R_k}$ is called the *impedance of the resistor* R_k (which in this case is also its resistance), $\underline{S_k/\omega i}$ is called the *impedance of the condenser* S_k, $\underline{i\omega L_k} = \underline{i\omega L_{kk}}$ is called the *self-impedance of the coil* L_k, and $\underline{i\omega L_{kl}}$ is called the *mutual impedance* of the coil *k* with the coil *l* (or between them). We then see from eqs. (4) that the complex voltages drops in a resistance and in an elastance are the corresponding impedances multiplied by the complex currents through them; and the self- and mutual-complex voltage drops in an inductance are equal to the corresponding self- and mutual-impedances multiplied by the corresponding complex currents.

It can also be noticed that the last equations of the system (3), for the generic node *n* and mesh *m*, are exactly of the same form as the corresponding Kirchhoff's Laws for instantaneous values but with the latter replaced by complex numbers. Therefore they will be call- *Kirchhoff's complex* (current and voltage) *laws*. The second of the equations of the system (3) is *Kirchhoff's Complex Current Law* for the generic node *n*. It is also known as *Kirchhoff's Vector*, or *Symbolic, Current Law*, in the literature. It states that the (algebraic) sum of all the complex currents leaving (or entering) a node is zero. The last equation of the system (3) is *Kirchhoff's Complex, Vector*, or *Symbolic, Law* for the generic mesh *m*, and it states that the (algebraic) sum of all the complex voltage drops (or rises) taken in a given sense around a mesh is zero.

The first of the equations of the system (3), giving the complex voltage drop in the generic element *k*, will be called the *complex voltage equation* for the generic element *k*.

If in any given network we replace each circuit parameter by the corresponding impedance, and every current and voltage (whether an exciting or a response quantity) by the corresponding complex current and voltage, respectively, we obtain the corresponding <u>transformed network</u>. The transformed network will have the same graph as the given network and therefore the same combinatorial structure. In the transformed network there are only complex numbers, and time functions no longer appear; nevertheless, in practice, <u>hybrid networks</u> are frequently used in which some quantities are transformed (usually the currents and voltages) while others are not (usually the parameters); but it is usually easy to obtain the transformed network from any hybrid network, so that this is unimportant. A typical general series element of a transformed network is shown in Fig. 2; this is the transform of the element shown in Fig. 1.

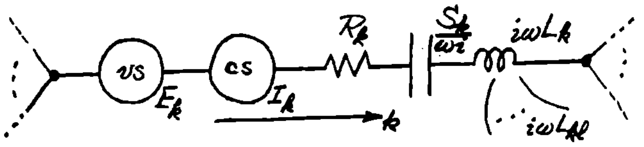


Fig. 2.

<u>Example 1</u>. Consider the network shown in Fig. 3. The transformed network is shown in Fig. 4. Assume $E^*=125\sqrt{2}\sin(100\pi t + 10°)$.

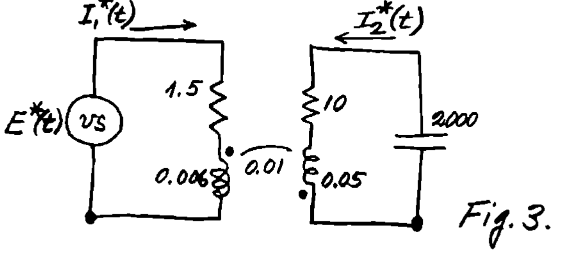


Fig. 3.

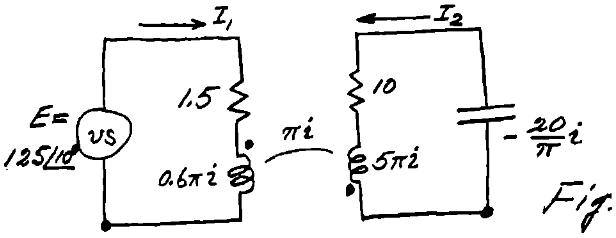


Fig. 4.

All the values given are in MKSC-units, the value given for the mutual inductance being its absolute value only; in fact, according to the polarity marks shown, the mutual inductance referred to the assigned reference directions is -0.01. The network has two elements of the general series type; there are two nodes and two components so that there are zero independent nodes, and no Kirchhoff's Current Laws are needed. There are two independent meshes, and so two Kirchhoff's Voltage Laws are needed which, in this case, are simply: $V_1^*=0$ and $V_2^*=0$, of which the transformed equations are: $V_1=0$ and $V_2=0$.

The equations of the original network are the following:

$$\begin{aligned} V_1^* &= 1.5\, I_1^* + 0.006\, dI_1^*/dt - 0.01\, dI_2^*/dt - E^*(t) = 0,\\ V_2^* &= 10\, I_2^* + 2000\int I_2^*\, dt + 0.05\, dI_2^*/dt - 0.01\, dI_1^*/dt = 0, \end{aligned} \tag{5}$$

and the transformed equations (which may be considered as the equations of the transformed network) are the following (taking $K = 1/\sqrt{2}$):

$$\begin{aligned} V_1 &= 1.5\, I_1 + 0.006\,(100\pi i) I_1 - 0.01(100\pi i) I_2 - 125\angle 10° = 0,\\ V_2 &= 10\, I_2 + 2000\, I_2/(100\pi i) + 0.05(100\pi i) I_2 - 0.01(100\pi i) I_1 = 0. \end{aligned} \tag{6}$$

These transformed equations can be written as follows:

$$(1.5 + 1.88495\,i)\,I_1 - 3.14159\,i\,I_2 = 125\underline{/10^\circ},$$
$$-3.14159\,i\,I_1 + (10 + 9.34175\,i)\,I_2 = 0,$$

of which the solution is:

$$\begin{aligned} I_1 &= 46.25539 - 21.06757\,i = 50.83\underline{/-24.53^\circ}, \\ I_2 &= 10.78323 + 4.45812\,i = 11.67\underline{/22.46^\circ}, \end{aligned} \qquad (7)$$

as can easily be checked. These are the complex currents.

The sinusoidal currents in the original network are then:

$$\begin{aligned} I_1^* &= 50.83\sqrt{2}\,\sin(100\pi t - 24.53^\circ) = 71.88\,\sin(100\pi t - 24.53^\circ), \\ I_2^* &= 11.67\sqrt{2}\,\sin(100\pi t + 22.46^\circ), \text{ in ampères.} \end{aligned} \qquad (8)$$

It can be shown that the generalized natural frequencies (see footnote to the example of Ch. IV, §6, p. 110), or Prob. 21 (at the end of this §) of the network considered in this example are the following:

$$p_1 = -448.5, \qquad p_2 = -113.25 + 143.45\,i, \qquad p_3 = -113.25 - 143.45\,i,$$

and hence that the complete solution of the differential equations (5) is:

$$\begin{aligned} I_1^*(t) &= A_1 e^{-448.5t} + B_1 e^{-113.25t}\sin(143.45t + \beta_1) + 71.88\sin(100\pi t - 24.53^\circ), \\ I_2^*(t) &= A_2 e^{-448.5t} + B_2 e^{-113.25t}\sin(143.45t + \beta_2) + 16.50\sin(100\pi t + 22.46^\circ), \end{aligned} \qquad (9)$$

where A_1, A_2, B_1, B_2, β_1, β_2, are constants of integration dependent on the initial currents in the coils, and on the initial charge of the condenser, of the network; the first two terms in these expressions constitute the complementary functions of the eqs. (5), i.e. the solution of the eqs. (5) when $E^*(t)$ is replaced by 0, while the last terms constitute the particular solution of eqs. (5), which is the same as the sinusoidal solution (8), and which do not depend on the intial conditions of the network.

From the complete solution (9) it can be appreciated how fast the "transient" terms of the complementary functions die out, and after a few of the 50 per second cycles the complete solution practically reduces to the sinusoidal solution (8).

Once I_1^* and I_2^* have been found, the individual voltage drops in the basic elements of the network can be obtained by substituting I_1^* and I_2^* in the individual terms of the equations (5).

It may be noticed that the determination of the complex currents (7) was just an artifice used to obtain the actual sinusoidal currents (8). The actual currents in this case are a forward and back-

ward motion of electric charge, and the complex currents have no real physical meaning. Similarly, complex voltages have no real physical meaning; they are used as artifices to determine the actual sinusoidal voltages, which alternatingly change their polarities every half period.

Example 2. Consider now the simple circuit shown in Fig. 5, whose transformed network is shown in Fig. 6. The integro-differential equations are the following: $V^* = 0.02\frac{dI^*}{dt} + 10^4\int I^* dt - E^* = 0.$ (10)

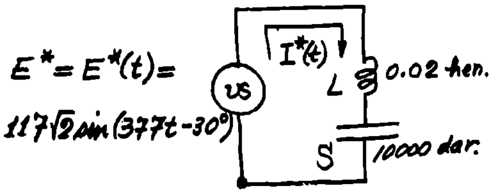


Fig. 5.

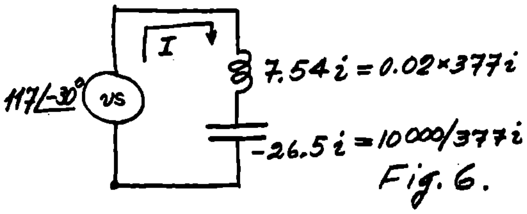


Fig. 6.

The transformed equations are (using: $K = 1/\sqrt{2}$*):*

$$V = 0.02(377\, i)I + 10000\, I/377\, i - 117\underline{/-30^\circ} = 0,$$

the solution of which is:

$$I = 6.16\underline{/\,60^\circ},$$

so that the sinusoidal current of angular frequency 377 (≅ 60 cps) is:

$$I^* = 6.16\sqrt{2}\,\sin(377\,t + 60^\circ) = 8.71\,\sin(377\,t + 60^\circ). \quad (11)$$

However, in this example it can be shown that the generalized natural frequencies (see Prob. 21) of the network are: $\pm\sqrt{S/L}\,i = \pm\,707.1\,i$ *, which do not have negative real parts, and so there may be other terms in the current which do not die out with time and hence are not ignorable. In fact, the general solution of the network of this example can be shown to be:*

$$I^*(t) = A\,\sin(707.1\,t + \alpha) + 8.71\,\sin(377\,t + 60^\circ), \quad (12)$$

where A and α *are integration constants which depend on the initial current,* I_0*, in the coil, and on the initial charge,* Q_0*, on the condenser, and, in this case, even on the EMF of the voltage source* E^**; specifically the complete solution is:*

$$I^*(t) = (I_0 - 7.55)\cos(707.1\,t) - (14.14\,Q_0 + 8.18)\sin(707.1\,t) + 8.71\sin(377t + 60^\circ). \quad (12')$$

From this equation it can be appreciated that even when the initial current I_0 *and initial charge* Q_0 *vanish, the total current differs considerably from the 60 ∿ sinusoidal current (11) found by the restricted alternating current (sinusoidal current) method.*

The situation would have been a little worse had the voltage source in the network of this example had an angular frequency ω equal to the natural angular frequency $\sqrt{S/L} = 1000/\sqrt{2} = 707.1$ sec.$^{-1}$ of the network. For in such a case, the restricted alternating current method would fail completely; and the general alternating current method would have to be used. In this way the current can be shown to be given by the following pe-function:

$$I^*(t) = (A' + B't - \omega Q_0)\sin \omega t + (I_0 + A'' + B''t)\cos \omega t\,,$$

where A', B', A'', B'' are constants depending only on the value of the voltage source (the exciting function) and Q_c and I_0 are constants of integration (the initial charge and current).

<u>Example 3</u>. Consider the network of Ch. III, §1, Fig. 2 (p. 65). Considering all the currents and voltages in the network to be sine functions of the same angular frequency ω, the transformed equations of the network can be obtained by substituting a complex number for each of the time functions in the general equations of the network and replacing differentiation and integration by multiplication and division by ωi, respectively. In this way we obtain the following transformed equations of the network in the sinusoidal state:

$$\begin{aligned}
V_1 &= R_1 I_1 + S_1 I_1/\omega i + L_1 \omega i I_1 + L_{13}\omega i I_3 - E_1\,,\\
V_2 &= R_2 I_2 + S_2 I_2/\omega i + L_2 \omega i I_2 + L_{24}\omega i I_4 + L_{28}\omega i I_8 - E_2\,,\\
V_3 &= R_3 I_3 + S_3 I_3/\omega i + L_3 \omega i I_3 + L_{31}\omega i I_1 + L_{37}\omega i I_7\,,\\
V_4 &= R_4 I_4 + S_4 I_4/\omega i + L_4 \omega i I_4 + L_{42}\omega i I_2 + L_{48}\omega i I_8\,,\\
V_5 &= R_5 I_5 + S_5 I_5/\omega i + L_5 \omega i I_5 + L_{59}\omega i I_9 - D_5\,,\\
V_6 &= R_6 I_6 + S_6 I_6/\omega i + L_6 \omega i I_6 + L_{6,10}\omega i I_{10} - D_6\,,\\
V_7 &= R_7 I_7 + S_7 I_7/\omega i + L_7 \omega i I_7 + L_{73}\omega i I_3\,,\\
V_8 &= R_8 I_8 + S_8 I_8/\omega i + L_8 \omega i I_8 + L_{84}\omega i I_4 + L_{82}\omega i I_2\,,\\
V_9 &= R_9 I_9 + S_9 I_9/\omega i + L_9 \omega i I_9 + L_{95}\omega i I_5\,,\\
V_{10} &= R_{10} I_{10} + S_{10} I_{10}/\omega i + L_{10} \omega i I_{10} + L_{10,6}\omega i I_6\,,\\
V_{11} &= R_{11} I_{11} + S_{11} I_{11}/\omega i\,,\\
V_{12} &= R_{12} I_{12} + S_{12} I_{12}/\omega i\,,
\end{aligned} \tag{13}$$

$$-I_3 + I_6 - I_7 + I_{10} + I_{12} = 0, \qquad I_3 + I_5 + I_7 + I_9 - I_{11} = 0, \qquad I_4 - I_5 + I_8 - I_9 - I_{12} = 0,$$

$$V_1 = 0, \quad V_2 = 0, \quad V_3 - V_7 = 0, \quad V_4 - V_8 = 0, \quad V_5 - V_9 = 0, \quad V_6 - V_{10} = 0,$$

$$V_7 + V_{10} + V_{11} = 0, \qquad V_9 + V_8 + V_{11} = 0, \qquad -V_{10} + V_{12} + V_8 = 0.$$

The transformed equations (3) of a network in the sinusoidal state can, of course, be established directly from the network; there is no real need of first establishing the original equations (1) and then transforming them according to the isomorphism (2). It is easy enough to write letters representing complex currents and voltages as it is to write letters representing sinusoidal currents and voltages of the same angular frequency ω, and to multiply and divide by ωi instead of differentiating and integrating, respectively, with respect to the time.

All one has to know is: that the complex voltage drops in the passive basic elements of a network are given by eqs. (4), p.117, that the complex voltage drops in the current and voltage sources are the negatives of their complex ~~values~~ voltage rises, that the complex voltage drop along any given path through elements of the network, in a given sense, is equal to the sum of the complex voltage drops through the traversed elements in the given sense along the path and, in particular, that this sum is zero when the path is closed (Kirchhoff's complex voltage law), and that the sum of all the complex currents leaving (or entering) any node is zero (Kirchhoff's complex current law). The equations obtained directly in this way may be considered as the equations of the transformed network.

Example 4. Consider the network shown in Fig. 7. The basic

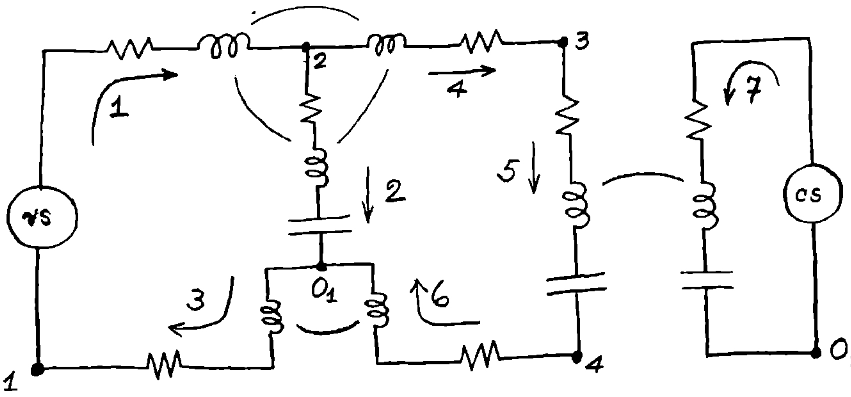


Fig. 7.

elements have been grouped into seven elements of the general series type, demarked by heavy dots in the figure, and oriented and numbered consecutively from 1 to 7. The only non-zero mutual inductances have been indicated by arcs, as usual. There are two components (=separate parts) in the network. Omitting one node in each component (O_1 and O_2), the rest were numbered consecutively from 1 to 4. There are 7-4=3 independent meshes in the network; for example, the meshes: 123, 2456, and the mesh formed by the element 7, form a complete and independent set of meshes (amongst other such sets), which may be taken as the meshes 1, 2, and 3, respectively.

As usual, let R_k, S_k, $L_k=L_{kk}$ denote the resistance, elastance, and self-inductance of the element $\underline{k}$, respectively, and let L_{kl} denote the mutual inductance between the (coils in the) elements $\underline{k}$ and $\underline{l}$ with respect to the assigned reference directions, all assumed to be constant. Also, considering the network to be in the sinusoidal state at the angular frequency ω, let V_k and I_k denote, as usual, the complex voltage drop and current through the element $\underline{k}$ in the assigned reference direction, and let E_k and D_k denote the complex voltage rises in the voltage and current sources of the element $\underline{k}$ in the assigned reference direction. Then the transformed equations of the network are:

$$\begin{aligned}
&V_1 = R_1I_1 + i\omega L_1I_1 + i\omega L_{12}I_2 + i\omega L_{14}I_4 - E_1,\\
&V_2 = R_2I_2 + S_2I_2/i\omega + i\omega L_2I_2 + i\omega L_{21}I_1 + i\omega L_{24}I_4,\\
&V_3 = R_3I_3 + i\omega L_3I_3 + i\omega L_{36}I_6,\\
&V_4 = R_4I_4 + i\omega L_4I_4 + i\omega L_{41}I_1 + i\omega L_{42}I_2,\\
&V_5 = R_5I_1 + S_5I_5/i\omega + i\omega L_5I_5 + i\omega L_{57}I_7,\\
&V_6 = R_6I_6 + i\omega L_6I_6 + i\omega L_{63}I_3,\\
&V_7 = R_7I_7 + S_7I_7/i\omega + i\omega L_7I_7 + i\omega L_{75}I_5 - D_7,\\
&I_1 - I_3 = 0, \quad \text{(at node 1)}\\
&-I_1 + I_2 + I_4 = 0, \quad \text{(at node 2)}\\
&-I_4 + I_5 = 0, \quad \text{(at node 3)}\\
&-I_5 + I_6 = 0, \quad \text{(at node 4)}\\
&V_1 + V_2 + V_3 = 0, \quad \text{(for mesh 1)}\\
&-V_2 + V_4 + V_5 + V_6 = 0, \quad \text{(for mesh 2)}\\
&V_7 = 0, \quad \text{(for mesh 3)}.
\end{aligned} \tag{14}$$

Problem 1. Establish directly the transformed equations for the following networks (of Fig. 8):

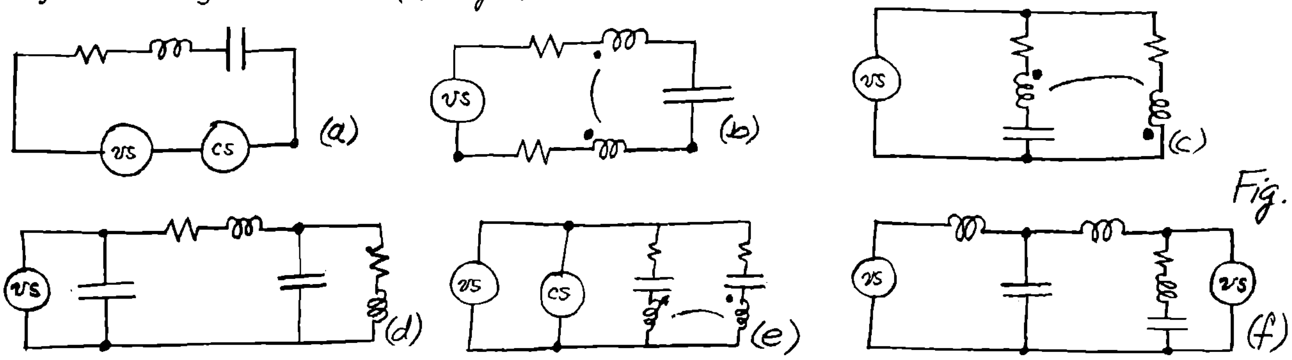


Fig. 8.

The first of the eqs. (3), p. 117, expressing the complex voltage drop in the generic element k of a network in terms of the complex currents through the elements, can be given a simpler appearance by introducing the quantities $Z_{kl} = Z_{kl}(i\omega)$ $(k,l=1,2,\dots,n_e)$, defined as follows:

$$Z_{kl} = \begin{cases} R_k + i\omega L_k + S_k/i\omega = R_k + i(\omega L_k - S_k/\omega), & \text{if } k = l, \\ i\omega L_{kl}, & \text{if } k \neq l. \end{cases} \tag{15}$$

We shall then have:

$$\begin{aligned} V_k &= R_k I_k + S_k I_k/i\omega + \sum_{l=1}^{n_e} i\omega L_{kl} I_l - E_k - D_k && (k=1,2,\dots,n_e) \\ &= (R_k + S_k/i\omega + i\omega L_k) I_k + \sum_{l\neq k} i\omega L_{kl} I_l - E_k - D_k && (\text{where } L_k = L_{kk}) \\ &= Z_{kk} I_k + \sum_{l\neq k} Z_{kl} I_l - E_k - D_k = \sum_{l=1}^{n_e} Z_{kl} I_l - E_k - D_k . \end{aligned}$$

The complete set of transformed equations of a network in the sinusoidal state will then appear in the final following form:

$$V_k = \sum_{l=1}^{n_e} Z_{kl} I_l - E_k - D_k, \qquad (k = 1,2,\dots,n_e) \tag{16}$$

$$\sum_{k=1}^{n_e} (k,n)\, I_k = 0, \qquad (n = 1,2,\dots,n_n' = n_n - n_c) \tag{17}$$

$$\sum_{k=1}^{n_e} [k,m]\, V_k = 0. \qquad (m = 1,2,\dots,n_m = n_e - n_n') \tag{18}$$

The quantities Z_{kl} introduced above are called the impedances of (and between) the elements of the network, at the angular frequency ω (or at the generalized frequency $0+\omega i$), a term due to O. Heaviside. Z_{kk} is frequently denoted simply by Z_k, and it is called the self-impedance of the element k. For $k \neq l$, Z_{kl} is called the mutual impedance between (or of) the elements k and l. (Z_{kk} may be considered as the mutual impedance of the element k with itself.) It can be noticed that these concepts agree with the previously introduced concepts of Impedances for the basic elements.

The imaginary part of Z_{kl}, usually denoted X_{kl}, is called the reactance (a term due to Hospitalier) between the elements k and l. We then have, by eqs. (15):

$$X_{kl} = \begin{cases} \omega L_k - S_k/\omega = \omega L_k - 1/\omega C_k, & \text{if } k = l, \\ \omega L_{kl}, & \text{if } k \neq l. \end{cases} \tag{19}$$

X_{kk}, usually denoted simply by X_k, is called the self-reactance, or simply reactance, of the element k. If $k \neq l$, X_{kl} is called the mutual reactance of, or between, the elements k and l.

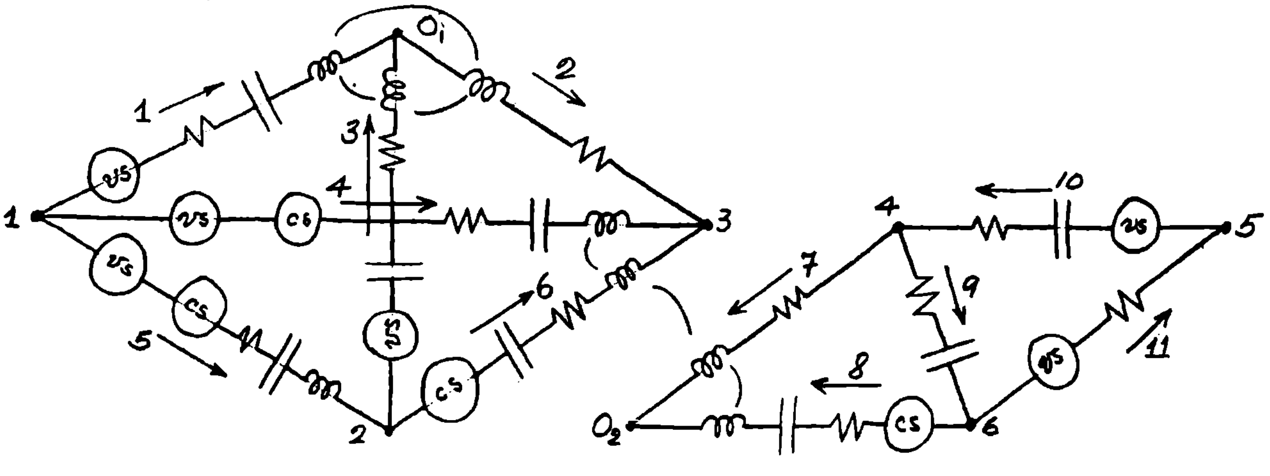
In a reactance such as: $X = \omega L - S/\omega = \omega L - 1/\omega C$, the term ωL, usually denoted by an expression such as X_L, is called the reactance of the corresponding coil, or the inductive reactance of the corresponding element; and the term $S/\omega = 1/\omega C$, usually denoted by an expression such as X_S or X_C, is called the capacitive reactance of the corresponding element, and $-S/\omega = -1/\omega C$ is called the reactance of the corresponding condenser (so that the adjective "capacitive" in the name of X_S or X_C takes the place of the negative sign). The reactance can then be written: $X = X_L - X_S = X_L - X_C$.

The (self) impedance of an element, such as $Z = R + i(\omega L - S/\omega)$, can be written $Z = R + iX = R + i(X_L - X_S) = R + i(X_L - X_C)$, and the mutual impedance between two distinct elements: $Z_{kl} = i\omega L_{kl}$, can be written $Z_{kl} = iX_{kl}$.

It should be noticed that the impedances and reactances (of and between the elements of a network) depend only on the angular frequency of the sources and on the passive parts of the elements (i.e. on the parameters) of the network

The terms $Z_{kl} I_l$ of the sum in the complex voltage eqs. (16) are called ZI-drops; the term $Z_{kk} I_k$ is called the self ZI-drop in the element k due to its own complex current I_k, and $Z_{kl} I_l$ $(l \neq k)$ is called the mutual ZI-drop in element k due to the complex current I_l in element l (all in the reference directions assigned to the elements). The complex voltage drop in the passive part of an element is then the sum of the self and mutual ZI-drops in that element; and the total complex voltage drop in the element is equal to the difference between the complex voltage drop in its passive part and the voltage rises in its sources.

Example 5. Consider the network shown in Fig. 9. This network has 11 elements of the general serie type which were numbered and oriented arbitrarily as shown. There are 8 nodes and 2 components;



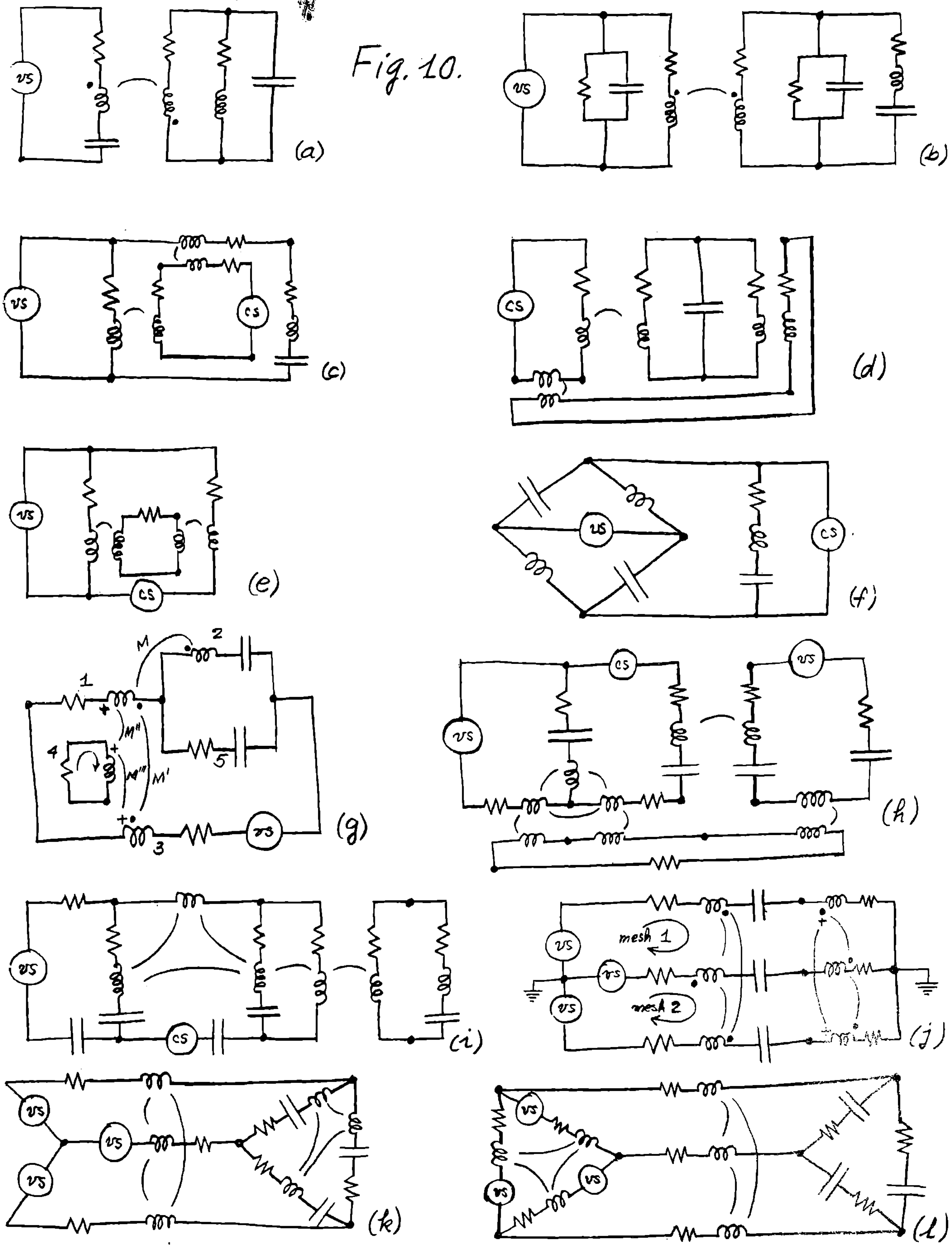
hence there are only 8-2=6 independent nodes. Omitting the nodes O_1 and O_2, the rest were numbered arbitrarily as shown. There are only 11-6=5 independent meshes, of which 6-3=3 are of the first component and 5-3=2 are of the second. These were chosen to be the meshes: $(2,\bar{4},1)$, $(6,\bar{4},5)$, $(1,\bar{3},\bar{5})$ in the first component, and $(10,9,11)$, $(7,\bar{8},\bar{9})$ in the second. The complete set of complex equations for the network in the sinusoidal state are the following:

$$\begin{gathered}
V_1 = Z_1 I_1 + Z_{12} I_2 + Z_{13} I_3 - E_1, \\
V_2 = Z_2 I_2 + Z_{21} I_1 + Z_{23} I_3, \\
V_3 = Z_3 I_3 + Z_{31} I_1 + Z_{32} I_2 - E_3, \\
V_4 = Z_4 I_4 + Z_{46} I_6 - E_4 - D_4, \\
V_5 = Z_5 I_5 - E_5 - D_5, \\
V_6 = Z_6 I_6 + Z_{64} I_4 + Z_{67} I_7 - D_6, \\
V_7 = Z_7 I_7 + Z_{76} I_6 + Z_{78} I_8, \\
V_8 = Z_8 I_8 + Z_{87} I_7 - D_8, \\
V_9 = Z_9 I_9 = (R_9 + S_9/i\omega) I_9, \\
V_{10} = Z_{10} I_{10} - E_{10} = (R_{10} + S_{10}/i\omega) I_{10} - E_{10}, \\
V_{11} = Z_{11} I_{11} - E_{11} = R_{11} I_{11} - E_{11}, \\
I_1 + I_4 + I_5 = 0, \\
I_3 - I_5 + I_6 = 0, \\
-I_2 - I_4 - I_6 = 0, \\
I_7 + I_9 - I_{10} = 0, \\
I_{10} - I_{11} = 0, \\
I_8 - I_9 + I_{11} = 0, \\
V_1 + V_2 - V_4 = 0, \\
-V_4 + V_5 + V_6 = 0, \\
V_1 - V_3 - V_5 = 0, \\
V_9 + V_{10} + V_{11} = 0, \\
V_7 - V_8 - V_9 = 0.
\end{gathered} \tag{20}$$

The unknowns in the above equations are the eleven complex voltage drops V_k $(k=1,2,\ldots,11)$,& I_1, I_2, I_3, I_7, I_9, I_{10}, I_{11}, D_4, D_5, D_6, and D_8. The known quantities are all the parameters (and hence all the impedances, Z_{kl}), all the complex electromotive forces E_1, E_3, E_4, E_5, E_{10}, E_{11}, of the voltage sources, and the complex currents I_4, I_5, I_6, I_8, through the current sources; and, of course, the angular frequency, ω, of all the currents and voltages.

Problem 2. Establish the complex equations for the following networks (Fig. 10), assumed to be in the sinusoidal state at the angular frequency $\omega \neq 0$:

Fig. 10.



The transformed equations (16), (17), and (18), for a network of n_e general series elements in the sinusoidal state at the angular frequency ω, form a (complete and independent) system of $2n_e$ linear algebraic equations in $2n_e$ unknowns (if the impedances and the complex currents and voltages are considered as indeterminates). The unknowns are all the complex voltage drops V_k, the complex currents I_k <u>not</u> belonging to current sources, and the complex voltage rises D_k in the current sources. All the complex electromotive forces E_k of the voltage sources, all the complex currents through the current sources, all the impedances Z_{kl}, and all the incidence numbers (k,n) and $[k,m]$ are considered as known, as well as ω.

In any particular case, assuming the system of equations to be consistent, the best way to solve the system for the unknowns is to substitute the complex voltage drops V_k, as given by the complex voltage equations (16), into the eqs. (18) expressing Kirchhoff's complex voltage laws. In this way the system of equations is reduced immediately to n_e equations in n_e unknowns. By means of eqs. (17), a further reduction in the number of equations and unknowns can be easily made by solving them for n'_n of the unknown complex currents and then eliminating them, in terms of the other complex currents. In this way the system of equations is reduced to $n_e - n'_n = n_m$ equations in n_m unknowns. By solving these n_m equations for the n_m unknowns left in them and then back-walking our previous steps, all the unknowns can be found. They can (and should) then be substituted into the $2n_e$ equations (16,17,&18) we began with, to check the solution.

Once all the unknown complex currents I_k and all the unknown complex voltages V_k and D_k have been found, the sinusoidal solution of the original integro-differential equations (1) can be obtained simply by using the isomorphism (2) in the reverse sense in order to find the corresponding unknown sinusoidal currents I_k^*, and the corresponding unknown sinusoidal voltages V_k^* and D_k^*, of angular frequency ω. Thus, if we have found that:

$$\begin{aligned} I_k &= a_k\underline{/\alpha_k} = |I_k|\underline{/\text{ang } I_k}\,, \\ V_k &= b_k\underline{/\beta_k} = |V_k|\underline{/\text{ang } V_k}\,, \\ D_k &= c_k\underline{/\gamma_k} = |D_k|\underline{/\text{ang } D_k}\,, \end{aligned} \tag{21}$$

then we obtain the corresponding sine functions of angular frequency ω, thus:

$$\begin{aligned} I_k^* &= (a_k/K)\sin(\omega t+\alpha_k) = (|I_k|/K)\sin(\omega t+\operatorname{ang} I_k), \\ V_k^* &= (b_k/K)\sin(\omega t+\beta_k) = (|V_k|/K)\sin(\omega t+\operatorname{ang} V_k), \\ D_k^* &= (c_k/K)\sin(\omega t+\gamma_k) = (|D_k|/K)\sin(\omega t+\operatorname{ang} D_k), \end{aligned} \qquad (22)$$

where, as we have said before, $K = 1/\sqrt{2}$ is usually taken, unless it is otherwise stated explicitly.

The final step of transforming the unknown complex numbers back to sine functions of the same angular frequency as that of all the sources is usually considered so obvious that it is usually omitted.

Problem 3. For the network of Fig. 7, considered in Ex. 4, p.123, show that if $V_1,\dots,V_7$, as given by the first seven of eqs. (14), are substituted into the last three, and then we eliminate I_2, I_3, I_5, & I_6, by substituting $I_2 = I_1 - I_4$, $I_3 = I_1$, $I_5 = I_4$, $I_6 = I_4$, according to Kirchhoff's complex current law, we obtain the equations:

$$\begin{gathered} \left[R_1+R_2+R_3+i\omega(L_1+L_{12}+L_2+L_{21}+L_3)+\frac{S_2}{i\omega}\right]I_1+\left[-R_4+i\omega(-L_2-L_{12}+L_{14}+L_{24}+L_{36})-\frac{S_2}{i\omega}\right]I_4 = E_1, \\ \left[-R_4+i\omega(-L_2-L_{21}+L_{41}+L_{42}+L_{63})-\frac{S_2}{i\omega}\right]I_1+\left[R_2+R_4+R_5+R_6+\frac{S_2+S_5}{i\omega}+i\omega(L_2-L_{24}+L_4-L_{42}+L_5+L_6)\right]I_4 = -i\omega L_{57} I_7, \\ D_7 = \left[R_7+i\omega L_7+\frac{S_7}{i\omega}\right]I_7+i\omega L_{75}I_4. \end{gathered} \qquad (23)$$

I_1 and I_4 can be found from the first two of these equations (E_1 being the complex value of a voltage source and I_7 that of a current source are consider known) and then D_7 can be found from the last. The other complex currents can be found in terms of I_1 and I_4 by the equations given above; and, finally, $V_1,\dots,V_7$ can then be obtained in terms of $I_1,\dots,I_7$, and D_7, by the first seven of eqs. (14).

Problem 4. Show that the unit of reactance is the same as that of resistance; hence, in the MKSC-system of units, the unit of reactance is the ohm. As a consequence, the units of the real and imaginary parts of an impedance are the same, and the same unit may be used for impedances; in the MKSC-system of units, then, an impedance is measured in ohms. However, some people prefer to use the terms "vector ohm", or "complex ohm", as units for impedances; but this is unnecessary and clumsy. Similarly, the units of the real and imaginary parts of a complex voltage are the same as that of a voltage; and the same can be said of currents. In the MKSC-system of units, then, complex voltages and currents may be measured in volts and ampères, respectively, although some people use the vector (or complex) volt, and the vector (or complex) ampere, respectively, as units for these quantities.

Problem 5. When all the L_{kl} are constant, we know that $L_{kl}=L_{lk}$ for all k and l. From this show that $Z_{kl} = Z_{lk}$ for all k and l.

Problem 6. Show that $X_{kl}^2 \leqq X_{L_k} \cdot X_{L_l}$ for all k and $l \neq k$, when all the L_{kl} are constant, as in a-c (alternating current) networks.

Problem 7. Show that if the reference direction of a single element k in an a-c network is changed then all the mutual impedances $Z_{kl} = Z_{lk}$ $(l \neq k)$ change signs (i.e. Z_{kl} $(l \neq k)$ becomes $-Z_{kl}$). Use this to show that the transformed equations of a network in the sinusoidal state remain essentially the same under a change of any number of the reference directions. This explains why the reference directions may be assigned arbitrarily.

Problem 8. Find the sinusoidal currents, of the same frequency as that of the source, in the elements of the network shown in Fig. 11. Also compute the partial voltage drops in the resistors and coils of the network by first finding the corresponding complex voltage drops by means of eqs. (4), p. 117. What would the sinusoidal currents and voltages be in this network if (only) instead of an angular frequency of 377 rad/sec we would have a frequency of 50 cps.

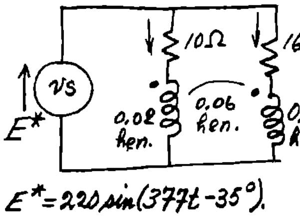


Fig. 11.

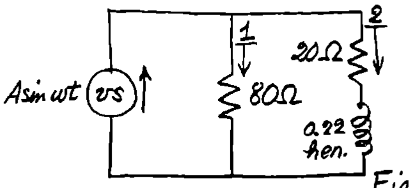


Fig. 12.

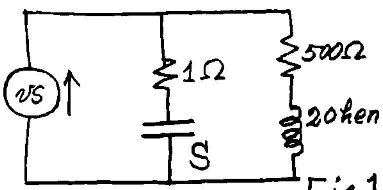


Fig. 13.

Problem 9. What must the frequency of the voltage source in the network shown in Fig. 12 be, in order that the branches 1 and 2 take the same effective current?

Problem 10. Compute the elastance (and the capacitance) of the condenser in the network shown in Fig. 13, so that the current through the voltage source be in phase with its electromotive force.

Problem 11. Consider the two elements connected in series as shown in Fig. 14. Let V_{12}, V_{23}, and V_{13} denote the complex voltage drops as indicated by the indices (in the double-subscript notation), and let I denote the complex current from 1 to 3, assuming the voltages and current to be sinusoids of the same frequency, of course. Let $R = R' = 0.001$ ohm, and let $X_L = X_C = 10$ ohms, while $|V_{13}| = 20$ volts. Show that $|I| = 10\,000$ amperes, and that $|V_{12}| = |V_{23}| \cong 100\,000$ volts. Notice that even though the effective voltage (as would be measured by a voltmeter) between the ends 1 and 3 is only 20 volts, that beween the points 1 and 2, and that between 2 and 3, are 10^5 volts, approx.

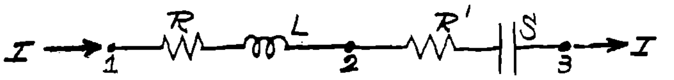


Fig. 14.

Thus a very high and dangerous voltage may exist between the intermediate point and the ends even though the total voltage from end to end is quite small. (Hence one must be careful in practice!) Compute the sinusoidal voltage drops V^*_{12} and V^*_{23} and see how they combine instantaneously to give the total sinusoidal voltage drop V^*_{13}, in order to get a better idea of the situation; and do the same thing with the corresponding complex voltage drops. It may be mentioned here that this is a simple example of what is known as (series-branch) current resonance and voltage anti-resonance (although some people refer to it as voltage resonance; see e.g. Malti, Circuit Theory, p.81); it is also called series-branch resonance, more specifically.

Problem 12. If $k \neq l$, show that the angle between the complex current I_l in element l and the mutual complex voltage drop $V_{k(l)} = Z_{kl}I_l$ it produces in the element k is 90°, I_l lagging if $X_{kl} > 0$ and I_l leading if $X_{kl} < 0$.

Problem 13. Show that the angle between (from) the complex current I_k in element k and (to) the self complex voltage drop $V_{k(k)} = Z_{kk}I_k = Z_kI_k$ it produces in it is equal to the angle of Z_k; that is:

$$\text{ang } V_{k(k)} - \text{ang } I_k = \text{ang } Z_k; \tag{24}$$

this angle shall be positive if and only if $X_k = X_{L_k} - X_{C_k} = \omega L_k - 1/\omega C_k > 0$, in which case the element k is said to be an inductive element, and the angle shall be negative if and only if $X_k = X_{L_k} - X_{C_k} = \omega L_k - 1/\omega C_k < 0$, in which case the element is said to be a capacitive element. When $X_k = X_{L_k} - X_{C_k} = \omega L_k - 1/\omega C_k = 0$, the element is said to be purely resistive. (Naturally, all this is said for a given angular frequency ω.)

Problem 14. In an a-c network, the charge on a condenser is also a pe-function and so in the restricted a-c problem, one may also speak of the sinusoidal charge Q^* and hence also of the corresponding complex, vector, or symbolic, charge Q on the condenser. Show that if the complex current through the condenser is I then the complex charge Q on the first plate of the condenser in the reference direction for the current is given by:

$$Q = I/i\omega, \qquad \text{and by} \qquad Q = C\,V_C = V_C/S, \tag{25}$$

where V_C is the complex voltage drop through the condenser in the reference direction for the current.

Problem 15. *In an a-c network the magnetic fluxes and flux linkages in the coils of the network are also pe-functions and so, in the restricted a-c problem, one may also speak of the sinusoidal magnetic fluxes ϕ_k^* and flux linkages Ψ_k^* and hence also of the corresponding complex, vector, or symbolic, fluxes ϕ_k and flux linkages Ψ_k in the coils k of the network. Show that if I_k are the complex currents in the coils, N_k their numbers of turns, and L_{kl} the inductances between the coils k and l, we shall have (cf. Ch. I, §3):*

$$\Psi_k = N_k \phi_k = \sum_l L_{kl} I_l = \frac{1}{\omega} \sum_l X_{kl} I_l = V_{L_k}/i\omega, \quad (\omega = \text{angular freq.} \ \& \ X_{kk} = \omega L_k) \tag{26}$$

where V_{L_k} is the complex voltage drop in the coil k. In particular, for an isolated coil (with all its mutual inductances with the other coils = 0) we shall have:

$$\Psi = N\phi = LI = X_L I/\omega, \tag{27}$$

where L is the (self) inductance and N is the number of turns of the coil.

Problem 16. *Suppose we have a system of n (idealized) coils with self and mutual inductances L_{kl} $(k, l = 1, 2, \ldots, n)$. Assume that all the coils are left open-circuited except coil l which is connected to a source (c.s. or v.s.) producing a current $I_l^* = A \sin(\omega t + \alpha)$ through it. Show that the transformed equations of the system are (denoting the complex voltage drops in the idealized coils by V_k):*

$$V_k = i\omega L_{kl} I_l, \qquad (k = 1, 2, \ldots, n)$$

and hence infer that: $|L_{kl}| = |V_k|/\omega|I_l|$. Thus, if an ammeter is inserted in coil l in order to measure the effective value $|I_l|$ of the current I^* through it, and a voltmeter is connected across the coil k $(=1, 2, \ldots, n)$ in order to measure the effective value $|V_k|$ of the voltage induced in it, then we can obtain the absolute value of the (self or mutual) inductance L_{kl} by dividing the reading $|V_k|$ of the voltmeter by ω times the reading $|I_l|$ of the ammeter (for each k and l).

Problem 17. *Assume we have a system of n sourceless series elements of the kind shown in Fig. 15. Let Z_{kl} $(k, l = 1, 2, \ldots, n)$ be the self and mutual impedances of, and between, these elements.*


Fig. 15.

Assume all these elements to be left open-circuited except element l which is connected to a source producing a current $I_l^* = A \sin(\omega t + \alpha)$ through it. Show that the complex voltage drops in the sourceless series elements are: $V_k = Z_{kl} I_l$, $(k = 1, 2, \ldots, n)$. From these equations we see that the modulus of each Z_{kl} can be obtained by dividing the reading $|V_k|$ of a voltmeter by the reading $|I_l|$ of an ammeter.

Of course, if the phase relations between the various voltage drops and the current in the element $\underline{l}$ are also determined (e.g. with an oscilloscope, or with a phase meter) then the impedances (moduli and angles) can be determined completely by: $Z_{kl} = V_k/I_l$ (for each $\underline{k}$ and $\underline{l}$). (When $k \neq l$ we know that $\arg Z_{kl} = \pm 90°$, of course.)

Problem 18. Show that the self impedance of a series element, of the kind shown in Fig. 15, can be determined completely by determining its modulus, $|Z|$, for three distinct frequencies. In particular, show that:

$$S = \lim_{\omega\to 0} \omega|Z| \qquad \text{and} \qquad L = \lim_{\omega\to\infty} |Z|/\omega\,. \qquad (\omega > 0)$$

(In practice, a constant voltage source could be used for the first of these and a sinusoidal source with a sufficiently high frequency could be used for the second.)

Problem 19. Show that in a network of N general series elements in the sinusoidal state, if a certain number M of the elements are connected in series, forming a branch with the same current through them, then the whole branch may be replaced by a single general series element with a self impedance $\tilde{Z}$ equal to the double (algebraic) sum of all the self and mutual impedances of and between the elements of the branch, and with a mutual impedance $\tilde{Z}_l$ with each of the other elements l of the network (not of the branch) equal to the (algebraic) sum of the mutual impedances of the element l in question with the elements of the branch. [Hint: Let the elements of the branch be numbered $\underline{1,2,\ldots,M}$, while the rest are numbered $\underline{M+1,M+2,\ldots,N}$, and let the elements of the branch be oriented in the same sense along the branch (re-orienting some of the elements, if necessary). Then if $\underline{V}$ denotes the total complex voltage drop along the branch in the reference direction common to all its elements and $\underline{I}$ denotes the common complex current through them, we shall have:

$$V = \sum_{k=1}^{M} V_k = \Big(\sum_{k=1}^{M}\sum_{l=1}^{M} Z_{kl}\Big)I + \sum_{l=M+1}^{N}\Big(\sum_{k=1}^{M} Z_{kl}\Big)I_l - \Big(\sum_{k=1}^{M} E_k\Big) - \Big(\sum_{k=1}^{M} D_k\Big) = \tilde{Z}I + \sum_{l=M+1}^{N} \tilde{Z}_l I_l - \tilde{E} - \tilde{D},$$

$$V_k = \Big(\sum_{l=1}^{M} Z_{kl}\Big)I + \sum_{l=M+1}^{N} Z_{kl} I_l - E_k - D_k = \tilde{Z}_k I + \sum_{l=M+1}^{N} Z_{kl} I_l - E_k - D_k\,, \quad (\text{for } k = M+1, M+2, \ldots, M)$$

while $\underline{V}$ can be substituted for $\sum_{k=1}^{M} V_k$ in Kirchhoff's Voltage Laws, and Kirchhoff's Current Laws for the intermediate nodes of the branch are already implied and can then be ignored. In particular we see that:

$$\tilde{Z} = \sum_{k=1}^{M}\sum_{l=1}^{M} Z_{kl} = \sum_{k=1}^{M} R_k + \frac{1}{i\omega}\sum_{k=1}^{M} S_k + i\omega\sum_{k=1}^{M}\sum_{l=1}^{M} L_{kl} = \tilde{R} + \frac{\tilde{S}}{i\omega} + i\omega\tilde{L}$$

$(\omega \neq 0$ [illegible]

$$\tilde{Z}_k = \sum_{l=1}^{M} Z_{kl} = \sum_{l=1}^{M} Z_{lk} = i\omega\sum_{l=1}^{M} L_{kl} = i\omega\tilde{L}_k\,, \quad (k = M+1, M+2,$$ [illegible]

and so the resistance $\tilde{R}$ and the elastance $\tilde{S}$ of the element replacing the branch must be equal, respectively, to the sum of the resistances in the branch and to the sum of the elastances in the branch. We also see that the complex EMF $\tilde{E}$ of the voltage source in the element replacing the branch must be equal to the sum $\sum_{k=1}^{M} E_k$, and that its current source must take care of a voltage rise $\tilde{\mathcal{D}}$ equal to the sum $\sum_{k=1}^{M} \mathcal{D}_k$ of the voltage rises in the current sources in the branch (which, of course, should all be of the same complex current $\underline{I}$, in order to have consistency to begin with).]

Problem 20. Show that the network of Fig. 7 may be reduced to the following network (Fig. 16), by first reducing the branch formed by the elements 1 and 3 to a single series element in accordance with the results of the preceding problem 19, and then doing the same with the branch formed by the elements 4, 5, and 6.

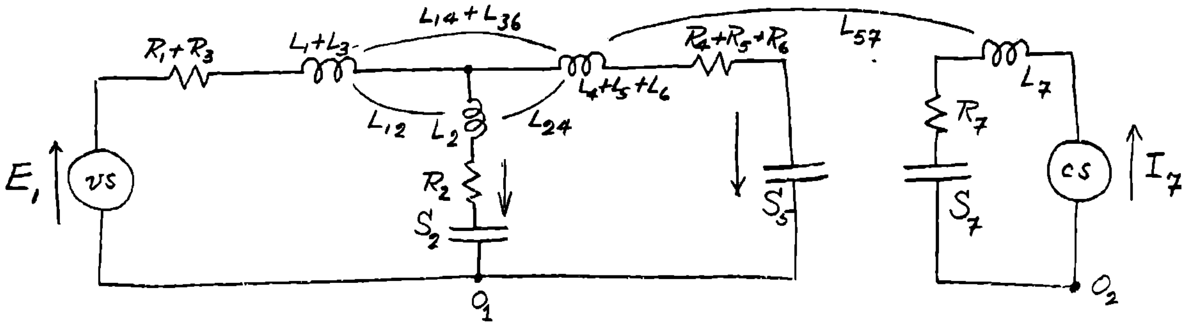


Fig. 16.

Problem 21. If $p=\sigma+\omega i$ is a generalized natural frequency of a network, the general equations of the network with all the (known) exciting functions nullified shall be satisfied by (unknown) response functions of the form $A \exp(pt)$, or of the form $A e^{\sigma t}\sin(\omega t+\alpha)$. Show that the generalized natural frequencies of an arbitrary (stationary) constant parameter network, with the general equations (3), (4), (5), of Ch. III, §1 (pp. 63-64), are given by the solutions, p, of the following determinantal equation (assuming that not every p is a solution):

$$
\begin{array}{c|cccc|cccc|ccc|}
\uparrow & Z_{11} & Z_{12} & \cdots & Z_{1N} & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
N & Z_{21} & Z_{22} & \cdots & Z_{2N} & 0 & -1 & \cdots & 0 & \vdots & & \vdots \\
 & \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots & \vdots & & \vdots \\
\downarrow & Z_{N1} & Z_{N2} & \cdots & Z_{NN} & 0 & - & - & -1 & 0 & \cdots & 0 \\
\hline
\uparrow & (1,1) & (2,1) & \cdots & (N,1) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
n'_n & (1,2) & (2,2) & \cdots & (N,2) & 0 & 0 & & 0 & \vdots & & \vdots \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\downarrow & (1,n'_n) & (2,n'_n) & \cdots & (N,n'_n) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
\uparrow & 0 & 0 & \cdots & 0 & [1,1] & [2,1] & - & - & - & - & [n_e,1] \\
n_m & 0 & 0 & \cdots & 0 & [1,2] & [2,2] & - & - & - & - & [n_e,2] \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & & & & \vdots \\
\downarrow & 0 & 0 & \cdots & 0 & [1,n_m] & [2,n_m] & - & - & - & - & [n_e,n_m] \\
\hline
\multicolumn{1}{c}{} & \multicolumn{4}{c}{\longleftarrow N \longrightarrow} & \multicolumn{4}{c}{\longleftarrow N \longrightarrow} & \multicolumn{3}{c}{\longleftarrow n_{cs} \longrightarrow}
\end{array} = 0,
$$

where N is the number of elements without current sources, and n_{cs} is the number of elements with current sources, assumed to be numbered last; while $Z_{kl}=(R_k+pL_k+S_k/p)$, if $k=l$, and $Z_{kl}=pL_{kl}$, if $k\neq l$.

(Hint: Make use of the results of Ch. IV, §7 and §6, and of a known result on the existence of non-trivial solutions of homogeneous linear equations.) (For a simpler equation, see Ch. VI, §4.

§2. NETWORKS OF GENERAL PARALLEL ELEMENTS IN THE SINUSOIDAL STATE.

If a network with many groups of basic elements of the kind shown in Fig. 1 were to be considered as a network of general series elements then there would result a large number of elements and meshes, and the method of the preceding section would be difficult to apply in practice to specific networks. In such a case it shall be appropriate to consider the network as one of general parallel elements.

Consider then an arbitrary (stationary) network of n_e general parallel elements of the kind shown in Fig. 1, arbitrarily oriented and numbered consecutively from 1 to n_e. Assume that these elements are connected into n_c components (=separate parts) with n_n nodes. Omitting exactly one node in each component, let the other nodes be arbitrarily numbered consecutively from 1 to $n_n' = n_n - n_c$. The components can be numbered consecutively from 1 to n_c, and the node omitted in the component $a(=1,2,\ldots,n_c)$, to be denoted 0_a, will be taken as the base or reference node in that component. Assume that a complete and independent set of $n_m = n_e - n_n' = n_e + n_c - n_n$ meshes has been chosen in the network (as explained at the end of Ch. II, §4, p. 60), and that they have been arbitrarily oriented and numbered consecutively from 1 to n_m; of course, in the process of choosing the meshes, each general parallel element must be considered as a two-terminal unit and the "interior" meshes must be ignored.

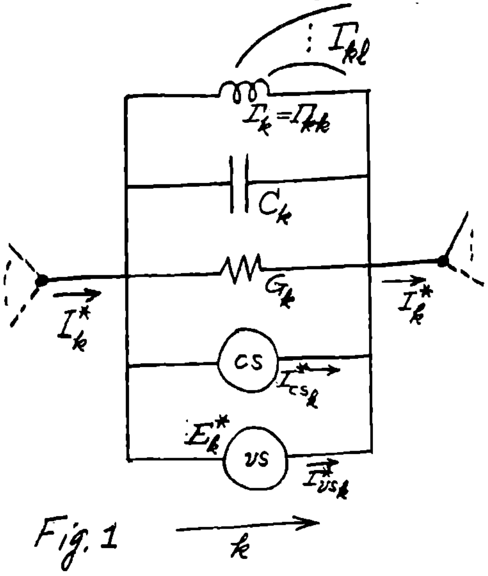


Fig. 1

Assume the network to be in the sinusoidal state, at the angular frequency $\omega \neq 0$. Let V_k^* and I_k^* denote the sinusoidal voltage drop and (total terminal) current, respectively, through the element k, and let the known sinusoidal current through the current source in the element k be denoted $I_{cs_k}^*$, and let the unknown sinusoidal current through the voltage source in the element k be denoted $I_{vs_k}^*$ (while the known sinusoidal voltage rise through it is denoted E_k^*), all in the reference direction arbitrarily assigned to the element k. Of course, for the elements which have voltage sources we have $V_k^* = -E_k^*$, so that the voltage drops in such elements are known.

The general equations (4), (5), and (6), of Ch. III, §2, now become:

$$I_k^* = G_k V_k^* + C_k\, dV_k^*/dt + \sum_{l=1}^{n_e} \Gamma_{kl} \int V_l^*\, dt + I_{cs_k}^* + I_{vs_k}^*, \quad (k=1,2,\ldots,n_e)$$

$$\sum_{k=1}^{n_e} (k,n)\, I_k^* = 0, \quad (n=1,2,\ldots,n_n'); \qquad \sum_{k=1}^{n_e} [k,m]\, V_k^* = 0, \quad (m=1,2,\ldots,n_m). \tag{1}$$

Let V_k, I_k, I_{cs_k}, I_{vs_k}, E_k $(k=1,2,\ldots,n_e)$ denote the complex numbers corresponding to the sine functions V_k^*, I_k^*, $I_{cs_k}^*$, $I_{vs_k}^*$, E_k^*, respectively, according to the correspondence:

$$s^* = s^*(t) = A \sin(\omega t + \alpha) \longleftrightarrow s = KA\, \underline{/\alpha}, \tag{2}$$

between the class of sine functions of angular frequency ω and the class of complex numbers (where $K = 1/\sqrt{2}$ will always be taken, unless explicitly mentioned otherwise, in order to have the modulus or absolute value of $\underline{s}$ equal to the effective value of $\underline{s^*}$).

Then, assuming (as usual in a-c networks) that all the circuit parameters G_k, C_k, and Γ_{kl} are constant, and remembering that differentiation is carried over into multiplication by ωi and that integration is carried over into division by ωi and that linear combinations and equations are preserved by the correspondence (2), the above equations (1) transform into the following equations:

$$I_k = G_k V_k + i\omega C_k V_k + \sum_{l=1}^{n_e} \Gamma_{kl} V_l / i\omega + I_{cs_k} + I_{vs_k}, \quad (k=1,2,\ldots,n_e)$$

$$\sum_{k=1}^{n_e} (k,n)\, I_k = 0, \quad (n=1,2,\ldots,n_n'); \qquad \sum_{k=1}^{n_e} [k,m]\, V_k = 0, \quad (m=1,2,\ldots,n_m). \tag{3}$$

These equations (3) are called the <u>transformed equations</u> of the network of general parallel elements in the sinusoidal state. They are also called the <u>complex</u>, <u>vectorial</u>, or <u>symbolic</u>, <u>equations</u> of an alternating current network of general parallel elements (in the sinusoidal state). All the terms of the transformed equations are complex numbers (corresponding to the actual sinusoidal currents and voltages according to the relation (2)); time functions and their derivatives and integrals no longer appear in them, and the complex numbers should not be mixed with the time functions. The use of the complex numbers is just an artifice to solve the equations (1) for the unknowns, by first transforming the eqs. (1) into the eqs. (3) according to the relation (2), then solving the eqs. (3) for the transforms of the unknowns, and then re-transforming the results back to find the original unknowns, again by the relation (2), but backwards.

V_k is called the <u>transformed</u>, <u>complex</u>, or <u>vector</u>, <u>voltage drop</u>

in the element k, I_k is called the (total) transformed, complex, or vector, (terminal) current through the element k, and I_{cs_k} and I_{vs_k} are called the transformed, complex, or vector, currents through the current and voltage sources, respectively, of the element k in the reference direction assigned to the element k, as was similarly mentioned in §1. The (known) complex electromotive forces E_k $(k=1,2,\ldots,n_e)$ of the voltage sources and the complex currents I_{cs_k} through the current sources of the network are called the complex exciting (or driving) quantities of the network (and of the corresponding transformed equations). The unknown complex currents and voltages are called the complex response quantities of the network and transformed equations.

The first of the equations (3) is called the transformed, complex, or vector, current equation for the generic element k, since it gives the (total terminal) complex current through it. The first three terms in this equation are the complex currents I_{G_k}, I_{C_k}, I_{Γ_k}, (say) through the resistor, condenser, and coil, respectively, of the element k, in the assigned reference direction; that is:

$$I_{G_k} = G_k V_k, \qquad I_{C_k} = i\omega C_k V_k, \qquad I_{\Gamma_k} = \sum_{l=1}^{n_e} \frac{\Gamma_{kl}}{i\omega} V_l, \quad (k=1,2,\ldots,n_e) \tag{4}$$

When the coil of the element k is magnetically isolated from the other coils of the network, the last of the eqs. (4) reduces, of course, to the equation: $I_{\Gamma_k} = \Gamma_k V_k / i\omega = V_k / i\omega L_k$; but in general, the complex current through the coil shall be a linear combination of all the complex voltage drops in all the coils of the network.

The second of the eqs. (3) is called Kirchhoff's transformed, complex, or vector, current law for the generic node n, and the last of the eqs. (3) is called Kirchhoff's transformed, complex, or vector, voltage law for the generic mesh m. It can be noticed, as was mentioned before, that Kirchhoff's complex laws have the same form as the corresponding laws for instantaneous values.

The coefficients of the complex voltage drops V_k $(k=1,2,\ldots,n_e)$ in the eqs. (4) (and in the first of the eqs. (3)) are called the admittances of the basic elements; specifically, G_k is called the admittance of the resistor (whose conductance is also G_k) in the element k, $i\omega C_k$ is called the admittance of the condenser C_k, and $\Gamma_k / i\omega = \Gamma_{kk} / i\omega$ is called the self-admittance of the coil Γ_k (in the presence of the other coils of the network) and $\Gamma_{kl} / i\omega$ is called the mutual-admittance between the coils (of the elements) k and l (in the presence of all the other coils of the network). According to eqs. (4) we can then say that the complex currents through a conductance, a capacitance, or a magnetically isolated coil, are equal *to the corresponding admittances multiplied by the complex voltage drops through them; and, calling $\Gamma_k V_k/i\omega$ the self-complex current through the coil k and $\Gamma_{kl} V_l/i\omega$ the mutual-complex current through the coil k due to the (voltage drop in the) coil l, the complex current through the coil k is equal to the sum of the self- and mutual-complex currents through it; the self- and mutual-complex currents being equal to the self- and mutual-admittances multiplied by the corresponding complex voltage drops.*

If in any given network we replace each circuit parameter by the corresponding admittance, and every current and voltage (whether known or unknown) by the corresponding complex current and voltage, respectively, we obtain the corresponding transformed network. The transformed network will have the same graph and the same combinatorial structure as the given network. A typical general parallel element of a transformed network is shown in Fig. 2, this being the transform of the element shown in Fig. 1.

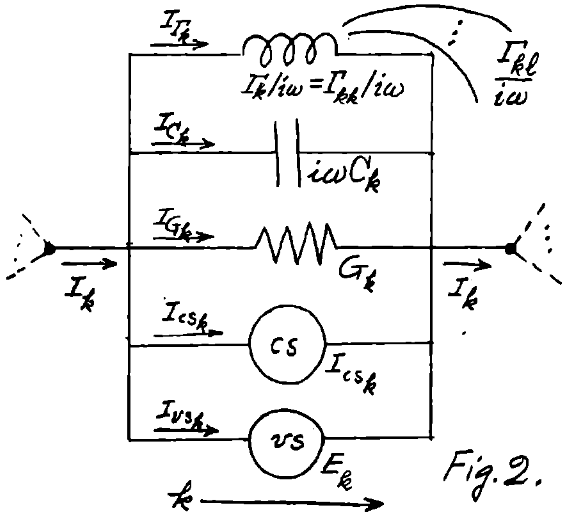


Fig. 2.

Of course, the transformed (complex) equations of a network in the sinusoidal state can be established directly from the network without having first to establish the original equations (1) and then to transform them according to the isomorphism (2). In fact, it is easier to establish the eqs. (3) than it is to establish the eqs. (1), by direct inspection of the network, or at least not harder.

All one needs to know to do this is: that the complex currents through the passive basic elements of a network are given by eqs. (4), that the complex current through a current source is known, that the complex current through a voltage source is unknown while its complex voltage rise (and drop) is known, that the sum of all the complex currents leaving (or entering) any node is zero (Kirchhoff's complex current law), and that the complex voltage drop along any given path through elements of the network, in a given sense, is equal to the sum of the complex voltage drops through the traversed elements in the given sense along the path and, in particular, that this sum is zero when the path is closed (Kirchhoff's complex voltage law). The equations obtained directly in this way may be considered as the original equations of the transformed network.

Example 1. Consider the network of Ch. III, §2, Fig. 2 (p. 73). Considering all the currents and voltages in the network to be sine functions of the same angular frequency ω, the transformed equations of the network can be obtained by substituting complex currents and voltages for the corresponding sinusoidal currents and voltages in the general equations of the network, and replacing differentiation and integration by multiplication and division by $i\omega$, respectively. In this way we obtain the following transformed equations of the network in the sinusoidal state (from the general equations given in the example of Ch. III, § 2, pp. 74-75, assuming the network parameters to be constant):

$$I_1 = I_{vs} + G_1V_1 + i\omega C_1V_1 + \Gamma_1V_1/i\omega + \Gamma_{12}V_2/i\omega + \Gamma_{15}V_5/i\omega,$$

$$I_2 = G_2V_2 + \Gamma_2V_2/i\omega + \Gamma_{21}V_1/i\omega + \Gamma_{25}V_5/i\omega,$$

$$I_3 = G_3V_3 + i\omega C_3V_3 + \Gamma_3V_3/i\omega + \Gamma_{37}V_7/i\omega,$$

$$I_4 = G_4V_4,$$

$$I_5 = \Gamma_5V_5/i\omega + \Gamma_{51}V_1/i\omega + \Gamma_{52}V_2/i\omega + \Gamma_{56}V_6/i\omega,$$

$$I_6 = G_6V_6 + \Gamma_6V_6/i\omega + \Gamma_{65}V_5/i\omega,$$

$$I_7 = I_{cs} + G_7V_7 + i\omega C_7V_7 + \Gamma_7V_7/i\omega + \Gamma_{73}V_3/i\omega,$$

$$I_8 = G_8V_8 + i\omega C_8V_8 + \Gamma_8V_8/i\omega.$$

$$-I_2 + I_3 = 0, \qquad -I_3 + I_4 = 0, \qquad -I_4 + I_5 = 0,$$

$$I_1 - I_5 = 0, \qquad I_6 = 0, \qquad I_7 + I_8 = 0.$$

$$V_1 + V_2 + V_3 + V_4 + V_5 = 0, \qquad V_7 - V_8 = 0.$$

Example 2. Consider the s-c (=sinusoidal current) network shown in Fig. 3(a). The basic elements have been grouped into eight elements of the general parallel type, as can better be appreciated from the graph (Fig. 3b).

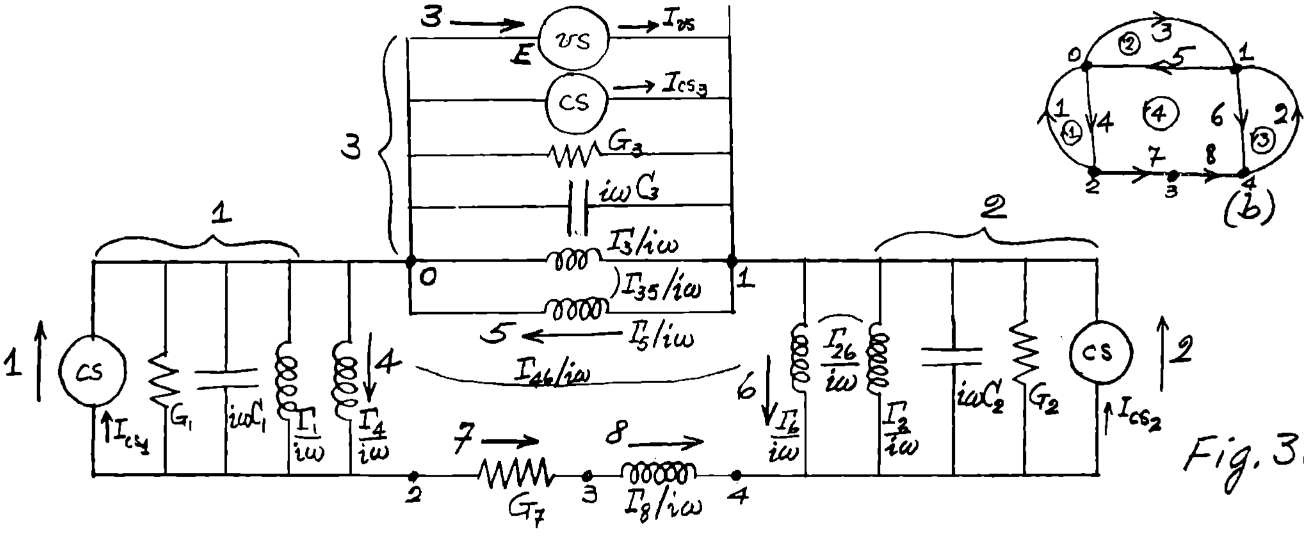


Fig. 3(a)

Notice that the elements between nodes 2 and 4 cannot be counted as a single element of the general parallel type. There is a single component and five nodes; hence there are four independent nodes. The node (whose Kirchhoff's current law is to be) omitted is marked 0, and the other nodes have been numbered arbitrarily 1,2,3,4. There are 8-4=4 independent meshes, as can readily be appreciated better from the graph of the network. Let us choose the following meshes: mesh 1: (1,4), mesh 2: (3,5), mesh 3: (2,6), mesh 4: $(4,7,8,\bar{6},5)$, oriented in the senses defined by the orientations of the leading (key) elements of the meshes when these have less than three elements. The elements have been arbitrarily oriented as shown by the arrows in Fig. 3(a); and the orientations assigned to the meshes are shown in the graph (b) by the curved arrows. Actually the network depicted in Fig. 3(a) is the transformed network, the equations of which are the following:

$$I_1 = I_{cs_1} + G_1V_1 + i\omega C_1V_1 + \Gamma_1V_1/i\omega,$$

$$I_2 = I_{cs_2} + G_2V_2 + i\omega C_2V_2 + \Gamma_2V_2/i\omega + \Gamma_{26}V_6/i\omega,$$

$$I_3 = I_{cs_3} + I_{vs} + G_3V_3 + i\omega C_3V_3 + \Gamma_3V_3/i\omega + \Gamma_{35}V_5/i\omega,$$

$$I_4 = \Gamma_4V_4/i\omega + \Gamma_{46}V_6/i\omega,$$

$$I_5 = \Gamma_5V_5/i\omega + \Gamma_{53}V_3/i\omega,$$

$$I_6 = \Gamma_6V_6/i\omega + \Gamma_{62}V_2/i\omega + \Gamma_{64}V_4/i\omega,$$

$$I_7 = G_7V_7,$$

$$I_8 = \Gamma_8V_8/i\omega;$$

$-I_2 - I_3 + I_5 + I_6 = 0$ (for node 1), $\quad I_1 - I_4 + I_7 = 0$ (for node 2),

$-I_7 + I_8 = 0$ (for node 3), $\quad I_2 - I_6 - I_8 = 0$ (for node 4);

$V_1 + V_4 = 0$ (for mesh 1), $\quad V_3 + V_5 = 0$ (for mesh 2),

$V_2 + V_6 = 0$ (for mesh 3), $\quad V_4 + V_7 + V_8 - V_6 + V_5 = 0$ (for mesh 4).

Of course, the I's and V's in these (transformed) equations of the network are complex currents and complex voltages, respectively. Also, the Γ's are the invertances referred to the assigned orientations of the elements. The unknowns in these equations are all the I_k and V_k $(k=1,2,\ldots,8)$, except V_3 (which is equal to $-E$ and so is known), and I_{vs}. Once these complex unknowns are found, the corresponding sinusoidal currents and voltages can be found by (2).

Example 3. Consider the network, assumed to be in the sinusoidal state at the angular frequency ω, shown in Fig. 4(a); the graph of which is shown in (b) in the usual fashion, and in (c) by identifying the terminals.

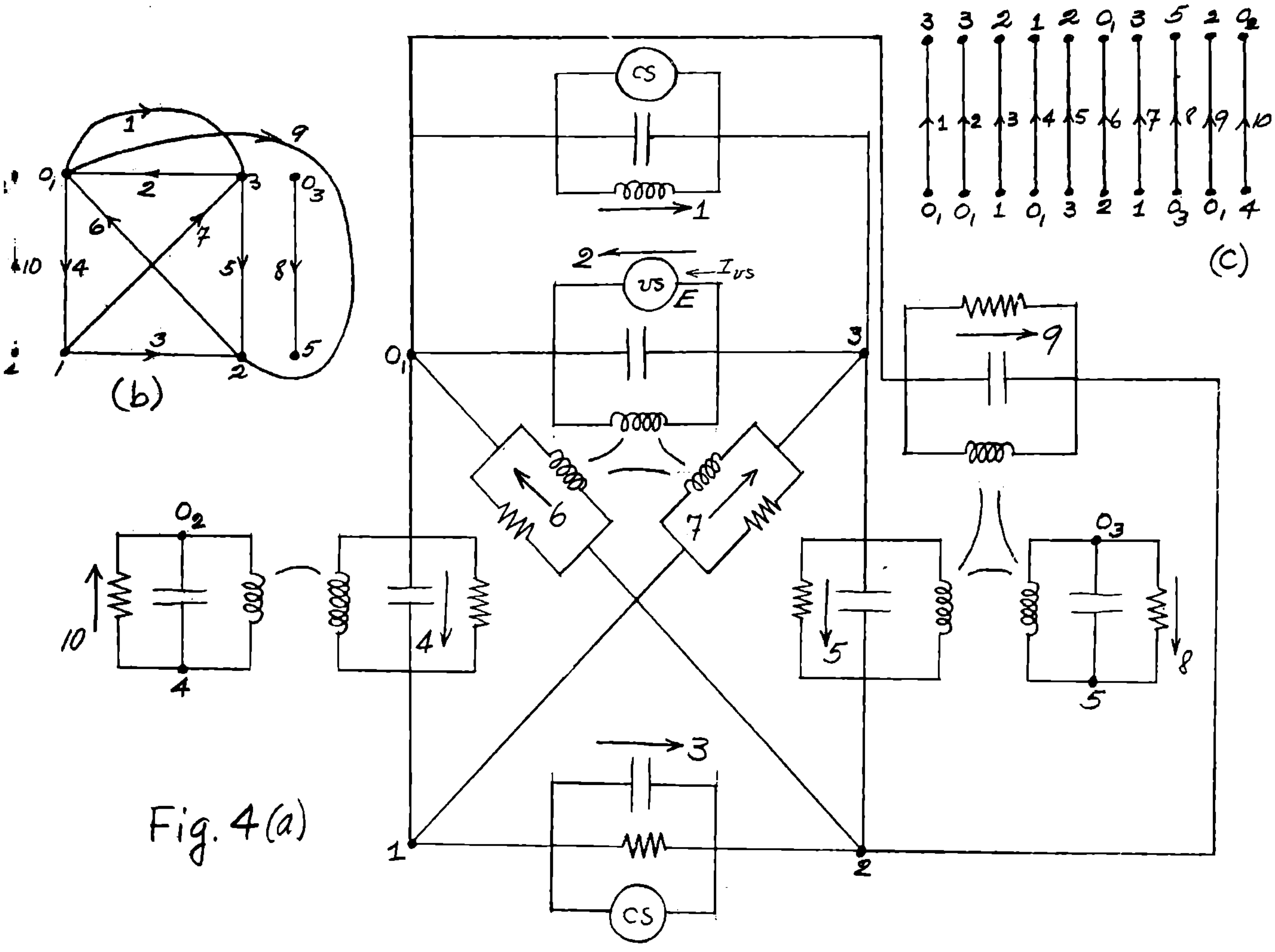


Fig. 4(a)

This network has three separate parts (components), eight nodes, and the basic elements have been grouped into ten elements of the general parallel type. Therefore, there are 8—3=5 independent nodes and 10-5=5 independent meshes. The elements have been arbitrarily oriented and numbered as shown in the figure. Omitting the node 0_1, 0_2, & 0_3, one in each component, the rest were numbered as shown. Let us take the following complete & independent set of meshes (chosen by the procedure given at the end of Ch. II, §4, p. 60): mesh 1 = (1,2), mesh 2 = (9,6), mesh 3 = ($\bar{2}$,5,6), mesh 4 = (4,3,6), mesh 5 = (5,$\bar{3}$,7). (See the graph (b))

The transformed equations of the network are the following:

$$I_1 = i\omega C_1 V_1 + \Gamma_1 V_1/i\omega + I_{cs_1},$$

$$I_2 = i\omega C_2 V_2 + \Gamma_2 V_2/i\omega + \Gamma_{26} V_6/i\omega + \Gamma_{27} V_7/i\omega + I_{vs} = \frac{\Gamma_{26}}{i\omega} V_6 + \frac{\Gamma_{27}}{i\omega} V_7 + I_{vs} - \left(i\omega C_2 + \frac{\Gamma_2}{i\omega}\right)E,$$

$$I_3 = i\omega C_3 V_3 + G_3 V_3 + I_{cs_3},$$

$$I_4 = G_4 V_4 + i\omega C_4 V_4 + \Gamma_4 V_4/i\omega + \Gamma_{4,10} V_{10}/i\omega,$$

$$I_5 = G_5 V_5 + i\omega C_5 V_5 + \Gamma_5 V_5/i\omega + \Gamma_{58} V_8/i\omega + \Gamma_{59} V_9/i\omega,$$

$$I_6 = G_6 V_6 + \Gamma_6 V_6/i\omega - \Gamma_{62} E/i\omega + \Gamma_{67} V_7/i\omega,$$

$$I_7 = G_7 V_7 + \Gamma_7 V_7/i\omega - \Gamma_{72} E/i\omega + \Gamma_{76} V_6/i\omega,$$

$$I_8 = G_8 V_8 + i\omega C_8 V_8 + \Gamma_8 V_8/i\omega + \Gamma_{85} V_5/i\omega + \Gamma_{89} V_9/i\omega,$$

$$I_9 = G_9 V_9 + i\omega C_9 V_9 + \Gamma_9 V_9/i\omega + \Gamma_{95} V_5/i\omega + \Gamma_{98} V_8/i\omega,$$

$$I_{10} = G_{10} V_{10} + i\omega C_{10} V_{10} + \Gamma_{10} V_{10}/i\omega + \Gamma_{10,4} V_4/i\omega;$$

$$I_3 - I_4 + I_7 = 0 \text{ (for node 1)}, \qquad -I_3 - I_5 + I_6 - I_9 = 0 \text{ (for node 2)},$$

$$-I_1 + I_2 + I_5 - I_7 = 0 \text{ (for node 3)}, \qquad I_{10} = 0 \text{ (for node 4)}, \quad -I_8 = 0 \text{ (for node 5)};$$

$$V_1 + V_2 = V_1 - E = 0 \text{ (for mesh 1)}, \quad V_9 + V_6 = 0 \text{ (for mesh 2)}, \quad -V_2 + V_5 + V_6 = 0$$

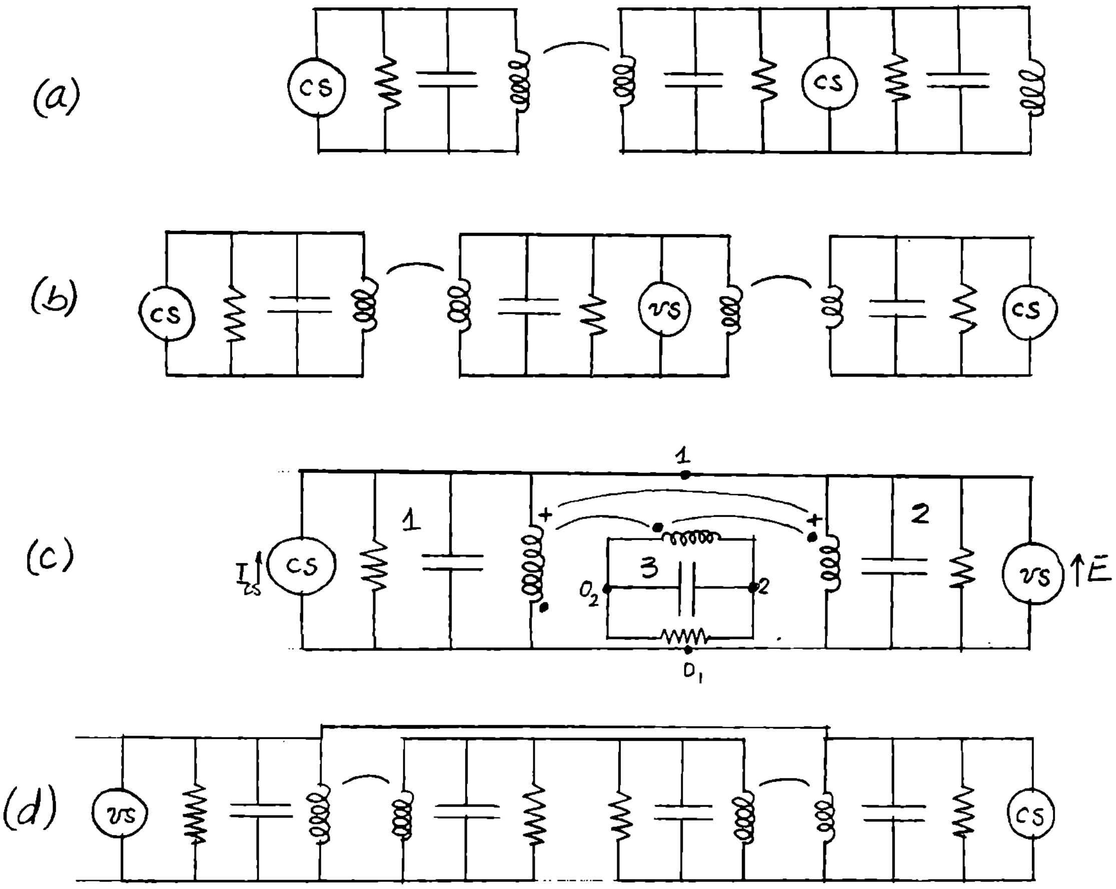
$$\text{or} \quad E + V_5 + V_6 = 0 \text{ (for mesh 3)}, \quad V_4 + V_3 + V_6 = 0 \text{ (for mesh 4)}, \quad V_5 - V_3 + V_7 = 0 \text{ (for mesh 5)}.$$

In these equations, G_k, C_k, and Γ_k, denote (as usual) the conductance, capacitance, and self-invertance of the element k, respectively, and Γ_{kl} denotes the mutual invertance between the (coils in the) elements k and l, referred to the assigned reference directions; all assumed to be constant, of course. The unknowns in these equations are the complex currents I_1, $I_2, \ldots, I_{10}, I_{vs}$, (although $I_8 = I_{10} = 0$ are already solved for, by Kirchhoff's current law applied to the nodes 4 and 5), and the complex voltages $V_1, V_3, V_4, \ldots, V_{10}$. The exciting quantities are E, I_{cs_1}, and I_{cs_3}, which are (assumed) known.

By substituting the complex currents as given by the first ten of the above equations into the next five (Kirchhoff's current laws), and then eliminating (for example): $V_1 = E$, $V_9 = -V_6$, $V_5 = -V_6 - E$, $V_4 = -V_6 - V_3$, $V_7 = V_3 - V_5 = V_3 + V_6 + E$, in accordance with the last five of the above equations (Kirchhoff's voltage laws), we can obtain five equations with five unknowns (I_{vs}, V_3, V_6, V_8, and V_{10}); with I_{vs} appearing in only one of the equations. By solving the latter reduced system and then reversing the previous steps, all the unknown complex currents and voltages can be found. Finally, the sinusoidal currents and voltages can easily be obtained by the isomorphism (2).

Problem 1. Establish the transformed equations for the networks of Figs. 8(d) and 10(b) of the preceding §1, assumed to be in the sinusoidal state at the angular frequency ω, but now by grouping the basic elements into general elements of the parallel type (in any convenient way) and using the parameters G, C, Γ's, instead of R, S, L's.

Problem 2. Establish directly the transformed equations for the following networks, assuming all the currents and voltages to be sinusoidal of the same angular frequency $\omega \neq 0$: (Fig. 5)



Figs. 5.

The first of the eqs. (3), p. 137, expressing the (total terminal) complex current entering the generic element k of a network in terms of the complex voltage drops in the elements of the network, can be given a simpler appearance by introducing the quantities $Y_{kl} = Y_{kl}(i\omega)$ $(k, l = 1, 2, \dots, n_e)$, defined as follows:

$$Y_{kl} = \begin{cases} G_k + i\omega C_k + \Gamma_k / i\omega = G_k + i(\omega C_k - \Gamma_k/\omega), & \text{if } k = l. \\ \Gamma_{kl}/i\omega, & \text{if } k \neq l. \end{cases} \tag{E}$$

We shall then have:

$$\begin{aligned} I_k &= G_k V_k + i\omega C_k V_k + \sum_{l=1}^{n_e} \frac{\Gamma_{kl}}{i\omega} V_l + I_{cs_k} + I_{vs_k} \\ &= \left(G_k + i\omega C_k + \frac{\Gamma_k}{i\omega}\right) V_k + \sum_{l \neq k} \frac{\Gamma_{kl}}{i\omega} V_l + I_{cs_k} + I_{vs_k} \qquad (\text{where } \Gamma_k = \Gamma_{kk}) \\ &= Y_{kk} V_k + \sum_{l \neq k} Y_{kl} V_l + I_{cs_k} + I_{vs_k} \\ &= \sum_{l=1}^{n_e} Y_{kl} V_l + I_{cs_k} + I_{vs_k}. \end{aligned}$$

The complete system of transformed equations of a network of n_e general parallel elements in the sinusoidal state can then be put in the following final form:

$$I_k = \sum_{l=1}^{n_e} Y_{kl} V_l + I_{cs_k} + I_{vs_k}, \qquad (k = 1,2,\dots,n_e) \qquad (6)$$

$$\sum_{k=1}^{n_e} (k,n)\, I_k = 0\,, \qquad (n = 1,2,\dots,n'_n = n_n - n_c) \qquad (7)$$

$$\sum_{k=1}^{n_e} [k,m]\, V_k = 0\,, \qquad (m = 1,2,\dots,n_m = n_e - n'_n) \qquad (8)$$

The quantities $Y_{kl} = Y_{kl}(i\omega)$ introduced above by eqs. (5) are called the admittances of (and between) the elements of the network, at the angular frequency ω, or at the generalized frequency $0+i\omega = i\omega$. Y_{kk}, frequently denoted simply Y_k, is called the self-admittance of the element k (in the presence of all the other elements of the network); and Y_{kl}, for $k \neq l$, is called the mutual-admittance between (or the element k and (or with) the element l (in the presence of all the other elements of the network). (Y_{kk} may be considered as the mutual-admittance of the element k with itself.) It can be noticed that when the general element reduces to one of the basic elements, then the concept of admittance for the general element reduces to the previously introduced concepts for the basic elements.

The imaginary part of Y_{kl}, usually denoted B_{kl}, is called the susceptance between the elements k and l (although some autors use the negatives of these terms for this). We then have, by eqs. (5):

$$B_{kl} = B_{kl}(\omega) = \begin{cases} \omega C_k - \Gamma_k/\omega\ , & \text{if } k=l, \\ -\Gamma_{kl}/\omega\ , & \text{if } k \neq l. \end{cases} \qquad (9)$$

B_{kk}, usually denoted simply by B_k, is called the self-susceptance, or simply the susceptance, of the element k (in the presence of all the elements of the network); and B_{kl}, for $k \neq l$, is called the mutual-susceptance of, or between, the elements k and l (in the presence of all the elements of the network).

In a susceptance such as: $B = \omega C - \Gamma/\omega$, the term ωC, usually denoted by an expression such as B_C, is called the susceptance of the corresponding condenser or the capacitive susceptance of the corresponding element; and the term Γ/ω, usually denoted by B_Γ or B_L, is called the inductive susceptance of the corresponding element, and $-\Gamma/\omega = -B_\Gamma = -B_L$ is called the susceptance of the corresponding coil in the presence of all the coils of the network (so that the adjective "capacitive" in the name of B_Γ or B_L takes the place of the negative sign in the susceptance). We then have: $B = B_C - B = B_C - B_L$.

The (self) admittance of an element, such as $Y = G + i(\omega C - \Gamma/\omega)$, can then be written $Y = G + iB = G + i(B_C - B_\Gamma) = G + i(B_C - B_L)$; and the mutual-admittance between two distinct elements, $Y_{kl} = \Gamma_{kl}/i\omega$, can be written $Y_{kl} = i\,B_{kl}$.

It should be noticed that the admittances and susceptances (of and between the elements of a network) depend only on the parameters (i.e. on the passive parts of the elements), and on the angular frequency of the sources, of the network.

The terms $Y_{kl}V_l$ of the sum in the complex current eqs. (6) are called <u>YV-currents</u>; the term $Y_{kk}V_k = Y_kV_k$ is called the self <u>YV-current</u> in the element <u>k</u> due to its own complex voltage drop V_k, and the term $Y_{kl}V_l$ $(l \neq k)$ is called the <u>mutual YV-current</u> in the element <u>k</u> due to the complex voltage drop V_l in the element <u>l</u>, all in the reference directions assigned to the elements. The complex current through the passive part of an element is then the sum of the self and mutual YV-currents in that element; and the total terminal complex current entering (and leaving) the element is equal to the sum of the complex current through its passive part and the complex currents through its sources (its active part), all considered in the same reference direction.

<u>Example 4</u>. Consider the network shown in Fig. 6, assumed to be in the sinusoidal state at the angular frequency $\omega \neq 0$, in which all the parameters are constant (as usual) and only those coils facing each other have non-zero mutual invertances (and inductances).

The transformed equations are:

$$I_1 = Y_1V_1 + Y_{14}V_4 + I_{cs_1},$$
$$I_2 = Y_2V_2 + Y_{25}V_5 + I_{cs_2},$$
$$I_3 = Y_3V_3 + Y_{36}V_6 + I_{vs}, \quad (V_3 = -E)$$
$$I_4 = Y_4V_4 + Y_{41}V_1,$$
$$I_5 = Y_5V_5 + Y_{52}V_2,$$
$$I_6 = Y_6V_6 - Y_{63}E,$$
$$I_1 = I_2 = I_3 = 0$$
$$I_4 = I_5 = I_6$$
$$V_4 + V_5 + V_6 = 0$$

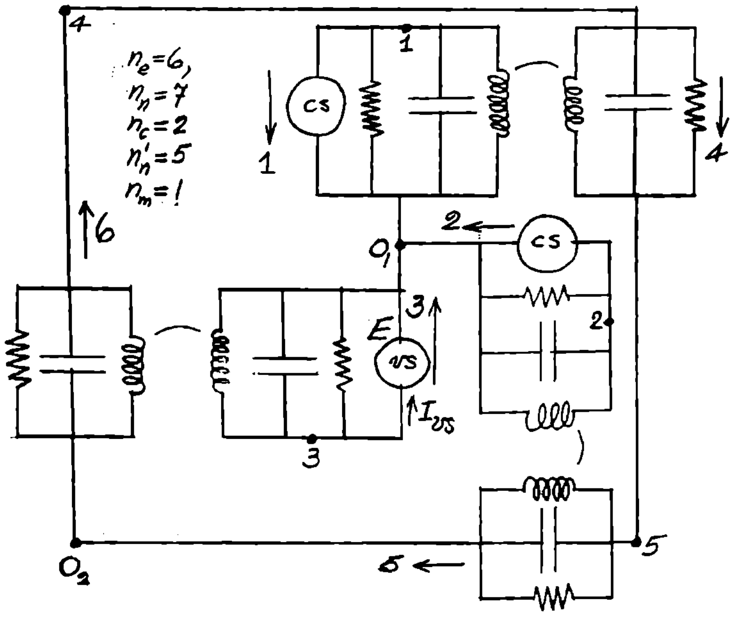


Fig. 6.

The known exciting quantities are: $E = -V_3$, I_{cs_1}, and I_{cs_2}.

The (a priori) unknown response quantities are: I_k, V_k $(k \neq 3)$, I_3, and I_{vs}.

Example 5. Consider the constant parameter $\overset{\text{s-c}}{\wedge}$ network of 18 general parallel elements whose graph is shown in Fig. 7, and assume that only the elements 1 to 4 are mutually coupled magnetically, and that only the elements 5 & 6 have sources. (Angular frequency $=\omega \neq 0$.)

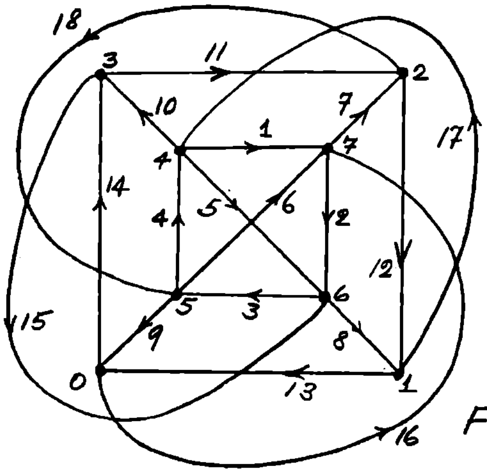


Fig. 7.

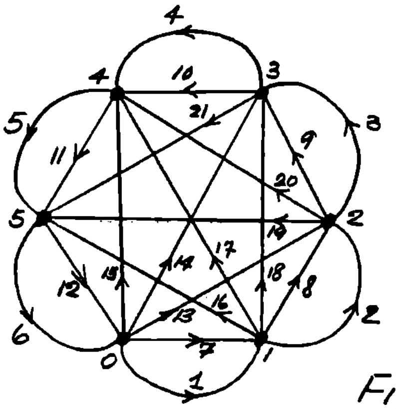


Fig. 8.

The network has 8 nodes and a single component; hence there are 8-1=7 independent nodes and 18-7=11 independent meshes. Let us take the following complete and independent set of meshes: (1,$\bar{6}$,4), (2,3,6), (3,4,5), (4,10,$\bar{14}$,$\bar{9}$), (5,$\bar{15}$,$\bar{10}$), (6,$\bar{16}$,$\bar{9}$), (7,18,9,16), (8,17,10,15), (9,$\bar{13}$,$\bar{12}$,18), (10,$\bar{14}$,$\bar{13}$,17), (11,12,13,14). The transformed equations are:

$$I_k=\sum_{l=1}^{4} Y_{kl}V_l \quad (k=1,2,3,4), \quad I_k=Y_kV_k+I_{cs_k}+I_{vs_k} \quad (k=5,6), \quad I_k=Y_kV_k \quad (k=7,8,\cdots,18);$$

$$I_{13}-I_8-I_{12}+I_{17}=0, \quad I_{12}-I_7-I_{11}+I_{18}=0, \quad I_{11}-I_{10}-I_{14}+I_{15}=0, \quad I_{10}-I_{17}+I_1+I_5-I_4=0,$$

$$I_4+I_6-I_3+I_9-I_{18}=0, \quad I_3-I_5-I_2+I_8-I_{15}=0, \quad -I_1+I_7-I_{16}+I_2-I_6=0;$$

$$V_1-V_6+V_4=0, \quad V_2+V_3+V_6=0, \quad V_3+V_4+V_5=0, \quad V_4+V_{10}-V_{14}-V_9=0,$$

$$V_5-V_{15}-V_{10}=0, \quad V_6-V_{16}-V_9=0, \quad V_7+V_{18}+V_9+V_{16}=0, \quad V_8+V_{17}+V_{10}+V_{15}=0,$$

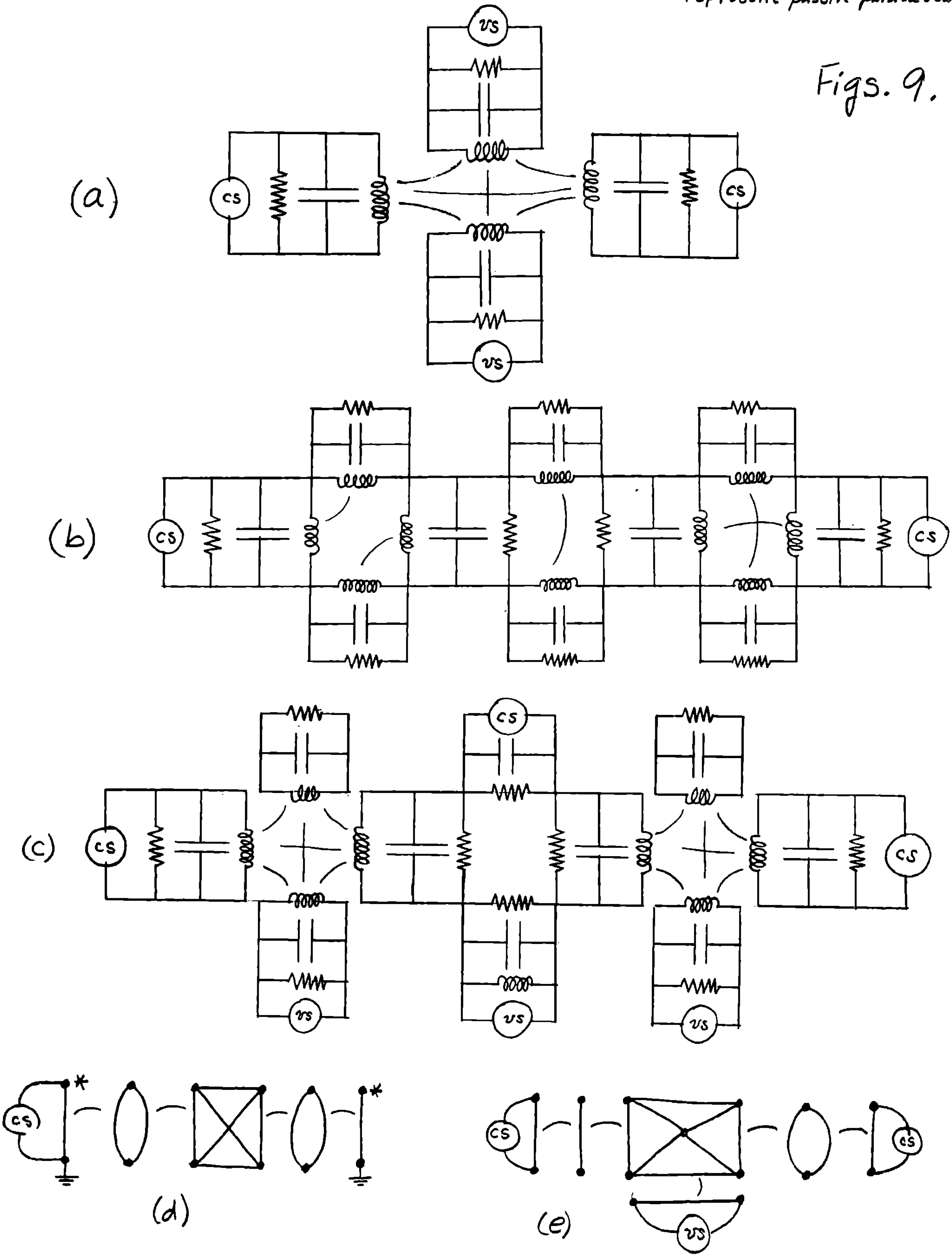
$$V_9-V_{13}-V_{12}+V_{18}=0, \quad V_{10}-V_{14}-V_{13}+V_{17}=0, \quad V_{11}+V_{12}+V_{13}+V_{14}=0.$$

Problem 3. Establish the transformed equations for the constant parameter sinusoidal-current network of 21 general parallel elements whose graph is shown in Fig. 8, assuming that all the elements have magnetically coupled coils and that there are no voltage sources in the network, but all the elements have current sources.

Problem 4. Show that the units of susceptances and admittances are the same as the unit of conductance. Hence, in the MKSC-system of units, susceptances and admittances are measured in mhos or inverse ohms (Ω^{-1}). However, in some of the older literature, the unit of admittance is called the vector-mho or complex-mho.

Problem 5. Establish the transformed equations for the following constant parameter networks (Fig. 9), assumed to be in the sinusoidal state at the angular frequency $\omega \neq 0$: (In (d) and (e) the line segments represent passive parallel elements.)

Figs. 9.



The transformed equations (6), (7) and (8), of a network of n_e general parallel elements in the sinusoidal state at the angular frequency ω, form a (complete and independent) system of $2n_e$ linear algebraic equations in $2n_e$ unknowns (if the admittances and the complex currents and voltages are considered as indeterminates). The unknowns are all the complex (terminal) currents I_k, the complex voltages V_k not belonging to elements with a voltage source, and the complex currents I_{vs_k} through the voltage sources. All the complex electromotive forces $E_k = -V_k$ (for the elements with voltage sources), all the complex currents I_{cs_k} through the current sources, all the impedances Z_{kl}, and all the incidence numbers (k,n) and $[k,m]$ are considered as known (as also the angular frequency ω, of course).

In any particular case, assuming the system of equations to be consistent, the best way to solve the system for the unknowns is to substitute the complex (terminal) currents I_k, as given by the complex current equations (6), into the eqs. (7) expressing Kirchhoff's complex current laws for the nodes of the network. In this way the system of equations is reduced immediately to n_e equations in n_e unknowns. By means of eqs. (8), a further reduction in the number of equations and unknowns can be easily made by solving them for n_m of the unknown complex voltage drops and then eliminating them, in terms of the other complex voltages. In this way the system of equations is reduced to $n_e - n_m = n'_n$ equations in n'_n unknowns. By solving these n'_n equations for the n'_n unknowns left in them and then back-walking our previous steps, all the unknowns can be found. They should then be substituted into the $2n_e$ equations (6,7,&8) we began with, to check the solution.

Once all the unknown complex currents I_k and I_{vs_k}, and all the unknown complex voltages V_k have been found, the sinusoidal solution of the original integro-differential equations (1) can be obtained simply by using the isomorphism (2) in the reverse sense in order to find the corresponding unknown sinusoidal currents I^*_k, I^*_{vs}, and voltages V^*_k, of angular frequency ω. Thus, if we have found that:

$$\begin{aligned} I_k &= a_k \underline{/\alpha_k} = |I_k| \underline{/\ \mathrm{ang}\ I_k}, \\ V_k &= b_k \underline{/\beta_k} = |V_k| \underline{/\ \mathrm{ang}\ V_k}, \\ I_{vs_k} &= c_k \underline{/\gamma_k} = |I_{vs_k}| \underline{/\ \mathrm{ang}\ I_{vs_k}}, \end{aligned} \tag{10}$$

then we obtain for the corresponding sinusoids of angular frequency ω:

$$
\begin{aligned}
I_k^* &= (a_k/K)\sin(\omega t+\alpha_k) = (|I_k|/K)\sin(\omega t+\operatorname{ang} I_k),\\
V_k^* &= (b_k/K)\sin(\omega t+\beta_k) = (|V_k|/K)\sin(\omega t+\operatorname{ang} V_k), \qquad (11)\\
I_{\upsilon s_k}^* &= (c_k/K)\sin(\omega t+\gamma_k) = (|I_{\upsilon s_k}|/K)\sin(\omega t+\operatorname{ang} I_{\upsilon s_k}),
\end{aligned}
$$

where, as we have said before, $K = 1/\sqrt{2}$ is usually taken, unless it is otherwise stated explicitly.

The final step of transforming the unknown complex currents and voltages back to sinusoidal currents and voltages, of the same angular frequency as that of all the sources, is usually omitted, as trivial.

Problem 6. Reduce the equations of the networks considered in the examples (1), (2), and (3), given above, to as many equations and unkowns as there are independent nodes in the corresponding networks.

Problem 7. When all the inductance L_{kl} are constant, we know that $L_{kl} = L_{lk}$, for all k and l. From this show that for all the invertances and admittances we have: $\Gamma_{kl} = \Gamma_{lk}$ and $Y_{kl} = Y_{lk}$.

Problem 8. Prove that: $B_{kl}^2 \leqq B_{\Gamma_k} \cdot B_{\Gamma_l}$, for all k and $l \neq k$. (Hint: See Prob. 13 of Ch. I, §3, p. 14.)

Problem 9. Show that if the reference direction of a single element k in an a-c network is changed then all the mutual admittances $Y_{kl} = Y_{lk}$ $(k \neq l)$ change signs (i.e. Y_{kl} changes into $-Y_{kl}$, if $l \neq k$). Use this to show that the transformed equations of a network in the sinusoidal state remain the same under a change of reference directions in any number of the elements. For this reason the reference directions may be assigned arbitrarily and we can be sure that we shall always obtain essentially the same system of equations.

Problem 10. Show that the complex charge Q_k (see Prob. 14, §1) on the first plate encountered in going through the condenser in the reference direction assigned to the element k is: $Q_k = C_k V_k$.

Problem 11. Show that the complex flux Φ_k (see Prob. 15, §1) in the coil of element k, in the associated reference direction, is given by: $\Phi_k = V_k / i\omega N_k$; and consequently that the complex flux linkages are given by: $\Psi_k = N_k \Phi_k = V_k / i\omega$.

Problem 12. If $k \neq l$, show that the angle between the complex voltage drop V_l in the element l and the mutual complex YV-current: $I_{k(l)} = Y_{kl} V_l$ it produces in the element k is 90°, with V_l lagging if $B_{kl} > 0$, and with V_l leading its mutual YV-current through element k if $B_{kl} < 0$, (i.e. if $\Gamma_{kl} > 0$ or $\Gamma_{lk} < 0$, respectively, assuming $\omega > 0$, of course).

<u>Problem 13</u>. *Show that the angle from the complex voltage drop, V_k, in the element $\underline{k}$, to its self complex current $I_{k(k)} = Y_{kk}V_k = Y_kV_k$, is equal to the angle of Y_k; that is:*

$$\text{ang } I_{k(k)} - \text{ang } V_k = \text{ang } Y_k; \tag{12}$$

this angle shall be positive if and only if $B_k = B_{C_k} - B_{\Gamma_k} = \omega C_k - \Gamma_k/\omega > 0$, in which case the element $\underline{k}$ is said to be a <u>capacitive element</u>; and it shall be negative if and only if $B_k = B_{C_k} - B_{\Gamma_k} = \omega C_k - \Gamma_k/\omega < 0$, in which case the element is said to be an <u>inductive element</u>, at the angular frequency ω. When $B_k = 0$, the element is said to be <u>purely conductive</u> (or resistive) at the angular frequency ω.

<u>Problem 14</u>. Assume that a current $\sqrt{2}\sin 377t$ is entering a passive element of the general parallel type. Let the conductance $\underline{G}$ of the element be 0.001 mhos, and let the capacitive and inductive susceptances of the element at the angular frequency 377 rad/sec be both equal to 10 mhos, i.e. $B_C = 377C = B_L = 1/(377L) = 10$, assuming the element to be magnetically isolated. Show that the complex voltage drop through the element is $1000\,\underline{/0^\circ}$ (volts). Thence show that the sinusoidal currents through the condenser and coil are, respectively:

$$I^*_C = 10\,000\sqrt{2}\cos 377t \qquad \& \qquad I^*_L = -10\,000\sqrt{2}\cos 377t.$$

Thus we see that, even though the current entering the parallel element is quite small, the currents in the condenser and coil (and the voltage drop through them) are very large (and so one must be careful practice). Here we have a simple example of what is called specifically <u>parallel-branch</u> (<u>voltage</u>) <u>resonance</u> and <u>parallel-branch current</u> <u>anti-resonance</u>.

<u>Problem 15</u>. Show that the complex currents through the condensers and coils of a given network of n_e general parallel elements are given by (see eqs. 4, for $k=1,2,\dots,n_e$):

$$I_{C_k} = i\,B_{C_k}V_k \qquad \& \qquad I_{\Gamma_k} = \sum_{l=1}^{n_e} i\,B_{kl}V_l. \quad (k=1,2,\dots,n_e) \tag{13}$$

<u>Problem 16</u>. Consider a system of $\underline{n}$ (idealized) coils with the self and mutual invertances (inverse inductances) Γ_{kl} (k & $l = 1,2,\dots,n$) referred to arbitrarily assigned orientattions (reference directions). Suppose that all the coils are short-circuited, except coil $\underline{l}$ which connected to a source producing a voltage drop $V^*_l = A\sin(\omega t + \alpha)$ across its terminals. Denoting the complex voltage drops and currents through the coils by V_k and I_k ($k = 1,2,\dots,n$), respectively, show that the complex (transformed) equations of the system are simply:

$$I_k = \Gamma_{kl} V_l / i\omega = \angle\alpha A\, \Gamma_{kl}/i\omega\sqrt{2}\,. \quad (k \,\&\, l = 1,2,\ldots,n)$$

From this we can then obtain the absolute values of the invertances:

$$|\Gamma_{kl}| = |\omega||I_k|/|V_k| = \sqrt{2}|\omega||I_k|/|A|\,. \quad (k \,\&\, l = 1,2,\ldots,n)$$

Hence, if an ammeter is inserted in coil $\underline{k}$ in order to measure the effective value $|I_k|$ of the induced current through it, and a voltmeter is connected to coil $\underline{l}$ in order to measure the effective value $|V_l| = |A|/\sqrt{2}$ of the applied voltage, then we can obtain the absolute value of the generic invertance Γ_{kl} by multiplying $|\omega|$ by the quotient of the readings: $|I_k|/|V_k|$.

Problem 17. Consider a system of $\underline{n}$ passive elements of the general parallel type, as the one shown in Fig. 10. Let Y_{kl} ($\underline{k}$ & $\underline{l}$ = 1,2,...,n) denote the self and mutual admittances of and between them, referred to arbitrarily assigned reference directions. Assume all these elements to be short-circuited (by connecting the terminals together as shown by the dotted line in the figure) except element $\underline{l}$ which is connected to a source producing a voltage drop $V_l^* = |A|\sin(\omega t+\alpha)$ across its terminals. Show that the (total) complex currents through the parallel elements are given simply by: $I_k = Y_{kl}V_l$ $(k = 1,2,\ldots,n)$ where, as usual, V_l denotes the complex voltage drop $(|A|/\sqrt{2})\underline{/\alpha}$. From these equations we see that all the Y_{kl} (for $\underline{k} = 1,2,\ldots,n$ and the given $\underline{l}$) can be determined experimentally by measuring the effective values $|I_k|$ and $|V_l|$ of, and the phase angles (ang I_k − ang V_l) between, I_k ($\underline{k} = 1,2,\ldots,n$) and V_l. By repeating the experiment for each $\underline{l} = 1,2,\ldots,n$, all the admittances Y_{kl} can be determined. In particular, the moduli, or absolute values, of the admittances can be obtained by: $|Y_{kl}| = |I_k|/|V_l|$, in terms of the effective values measured with an ammeter and a voltmeter (and, of course, for $k \neq l$ we know that the angle of Y_{kl} is ±90°).

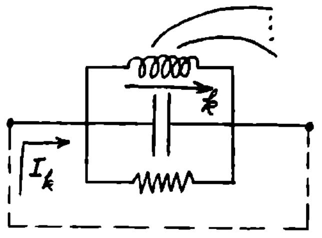


Fig. 10.

Problem 18. For the system of $\underline{n}$ passive elements of the general parallel type considered in the preceding problem 17, show that the self admittance $Y_k = Y_{kk}$ of element $\underline{k}$ (in the presence of all the other elements of the system) can be determined completely by determining its modulus $|Y_k|$ for three distinct frequencies. In particular, show that:

$$\Gamma_k = \lim_{\omega \to 0} \omega |Y_k| \qquad \text{and} \qquad C_k = \lim_{\omega \to \infty} |Y_k|/\omega\,. \qquad (\omega > 0)$$

(In practice, a constant source could be used for the first, and a sufficiently high frequency could be used for the second, of these.)

Problem 19. The resistances of the practical coils in a sinusoidal current network can not always be neglected in comparison with their associated reactances. It will then be convenient to study networks of general parallel "mixed" elements, of the kind shown in Fig. 11, in the sinusoidal state at the angular frequency $\omega \neq 0$. To do this, first consider a system of N practical coils with resistances R_k and self and mutual inductances L_{kl} (k & $l = 1,2,\ldots,N$), through which there are complex currents I'_k and complex voltage drops V_k, eferred to arbitrarily assigned reference directions. For this system we shall have:

$$V_k = R_k I'_k + \sum_{l=1}^{N} i\omega L_{kl} I'_l = \sum_{l=1}^{N} (R_k \delta_{kl} + i\omega L_{kl}) I'_l = \sum_{l=1}^{N} Z'_{kl} I'_l,$$

where $\delta_{kl} = 1$ if $k=l$ and $\delta_{kl} = 0$ if $k \neq l$, and $Z'_{kl} = R_k \delta_{kl} + i\omega L_{kl}$. Assuming $\Delta = \det(Z'_{kl}) \neq 0$, we can solve these equations for the complex currents I'_k in terms of the complex voltage drops V_k as follows:

$$I'_k = \sum_{l=1}^{N} Y'_{kl} V_l, \quad \text{where } Y'_{kl} = \operatorname{cof} Z'_{lk}/\Delta. \quad (k \,\&\, l = 1,2,\ldots,N) \tag{14}$$

Of course these Y'_{kl} can also be obtained experimentally in the manner explained in Prob. 17 (with all the condensers absent). With these results, and knowing that we have for the total complex terminal currents I_k entering the general parallel mixed elements:

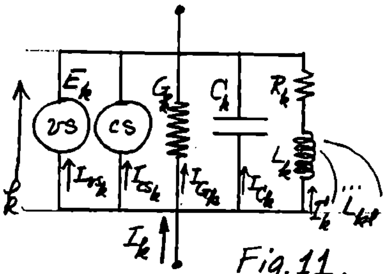


Fig. 11.

$$I_k = I_{vs_k} + I_{cs_k} + I_{G_k} + I_{C_k} + I'_k, \tag{15}$$

show that the general complex equations of an arbitrary network of N general parallel mixed elements connected in n_c components (separate parts) with n_n nodes and n_m independent meshes are:

$$\begin{aligned} I_k &= I_{vs_k} + I_{cs_k} + G_k V_k + i\omega C_k V_k + \sum_{l=1}^{N} Y'_{kl} V_l \\ &= I_{vs_k} + I_{cs_k} + \sum_{l=1}^{N} Y_{kl} V_l \; ; \qquad (k = 1,2,\ldots N) \end{aligned} \tag{16}$$

$$\sum_{k=1}^{N} (k,n)\, I_k = 0 \quad (n=1,2,\ldots,n'_n = n_n - n_c); \tag{17}$$

$$\sum_{k=1}^{N} [k,m]\, V_k = 0 \quad (m=1,2,\ldots,n_m = N - n'_n = N - n_n + n_c); \tag{18}$$

where we have put: $Y_{kl} = G_k + i\omega C_k + Y'_{kl}$ if $k=l$ and $Y_{kl} = Y'_{kl}$ if $k \neq l$. (19)

Problem 20. Show that in a network of N general parallel elements in the sinusoidal state, if a certain number M of the elements are connected in parallel, forming a parallel branch with the same voltage drop across them, then the whole parallel branch may be replaced by a single general parallel element with a self admittance $\tilde{Y}$ equal to the double (algebraic) sum of all the self and mutual admittances of and between the elements of the branch, and with a mutual admittance $\tilde{Y}_k$ with each of the other elements k of the network (not of the branch) equal to the (algebraic) sum of all the mutual admittances of the element k with the elements of the parallel branch. [Hint: Let the elements of the parallel branch be numbered last, with the numbers: $N-M+1 = \tilde{N}$(say), $\tilde{N}+1, \ldots, N$, and let the other elements of the network be numbered: $1, 2, \ldots, N-M$; also, let all the elements of the parallel branch be oriented in the same sense, reorienting some of the elements if necessary, and let V denote the common complex voltage drop across the elements of the branch and I the total terminal complex current entering the parallel branch. Then we have:

$$I_k = I_{vs_k} + I_{cs_k} + \sum_{l=1}^{N-M} Y_{kl} V_l + \Big(\sum_{l=\tilde{N}}^{N} Y_{kl}\Big) V = I_{vs_k} + I_{cs_k} + \sum_{l=1}^{N-M} Y_{kl} V_l + \tilde{Y}_k V, \qquad (k = 1, 2, \ldots, N-M)$$

$$I = \sum_{k=\tilde{N}}^{N} I_k = \sum_{k=\tilde{N}}^{N} I_{vs_k} + \sum_{k=\tilde{N}}^{N} I_{cs_k} + \sum_{l=1}^{N-M}\Big(\sum_{k=\tilde{N}}^{N} Y_{kl}\Big) V_l + \Big(\sum_{l=\tilde{N}}^{N}\sum_{k=\tilde{N}}^{N} Y_{kl}\Big) V = \tilde{I}_{vs} + \tilde{I}_{cs} + \sum_{l=1}^{N-M} \tilde{Y}_l V_l + \tilde{Y} V;$$

whereas I can be substituted for $\sum_{k=\tilde{N}}^{N} I_k$ in Kirchhoff's Current Laws, while Kirchhoff's Voltage Laws for the intermediate meshes of the parallel branch are already implied and can then be ignored. Comparing the above equations with those of a network of $\tilde{N} = N-M+1$ general parallel elements, we see that the last element $\tilde{N}$ replacing the parallel branch must have a self admittance $\tilde{Y}$ and mutual admittances $\tilde{Y}_k$ $(k = 1, 2, \ldots, N-M)$, and a current source of complex value $\tilde{I}_{cs}$, given by:

$$\tilde{Y} = \sum_{k=\tilde{N}}^{N}\sum_{l=\tilde{N}}^{N} Y_{kl}, \qquad \tilde{Y}_k = \sum_{l=\tilde{N}}^{N} Y_{kl} = \sum_{l=\tilde{N}}^{N} Y_{lk} \quad (k = 1, 2, \ldots, N-M), \quad \tilde{I}_{cs} = \sum_{k=\tilde{N}}^{N} I_{cs_k}, \tag{20}$$

while its voltage source, which must have the same voltage drop as all the voltage sources in the parallel branch, must (be able to) take care of a complex current equal to the sum of all the complex currents through the voltage sources of the parallel branch.]

Problem 21. Show that the network of Fig. 3(a), considered in the example 2 above, may be reduced to the following network (Fig. 12):

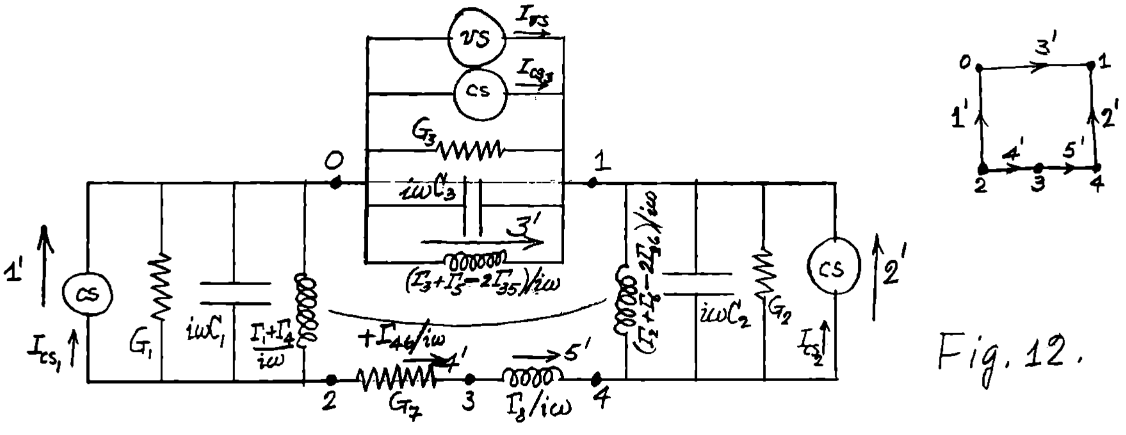


Fig. 12.

Problem 22. If $p=\sigma+i\omega$ is a generalized natural frequency of a network, the general equations of the network with all the exciting functions nullified shall be satisfied by response functions of the form $A\,e^{\sigma t}\sin(\omega t+\alpha)$, or of the form $A\exp(pt)$. Show that the generalized natural frequencies of an arbitrary (stationary) constant parameter network of general parallel elements, with the general equations (4), (5), (6), of Ch. III, §2 (p. 72), are given by the solutions, p, of the following determinantal equation (assuming that not every p is a solution, since otherwise this would be meaningless):

$$
\begin{array}{c|cccc:cccc:cccc|}
 & \multicolumn{4}{c}{N} & \multicolumn{4}{c}{N} & \multicolumn{4}{c}{n_{vs}} \\
\hline
 & Y_{11} & Y_{12} & \cdots & Y_{1N} & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
N & Y_{21} & Y_{22} & \cdots & Y_{2N} & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
 & Y_{N1} & Y_{N2} & \cdots & Y_{NN} & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \\
\hdashline
 & 0 & 0 & \cdots & 0 & (1,1) & (2,1) & \cdots & \cdots & \cdots & \cdots & \cdots & (n_e,1) \\
n_n' & 0 & 0 & \cdots & 0 & (1,2) & (2,2) & \cdots & \cdots & \cdots & \cdots & \cdots & (n_e,2) \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & & & & & \vdots \\
 & 0 & 0 & \cdots & 0 & (1,n_n') & (2,n_n') & \cdots & \cdots & \cdots & \cdots & \cdots & (n_e,n_n') \\
\hdashline
 & [1,1] & [2,1] & \cdots & [N,1] & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
n_m & [2,1] & [2,2] & \cdots & [N,2] & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots \\
 & [N,1] & [N,2] & \cdots & [N,N] & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\end{array}
\equiv
\begin{array}{c|c:c:c|}
 & N\text{ cols.} & N\text{ cols.} & n_{vs}\text{ cols.} \\
\hline
N\text{ rows} & \begin{matrix} Y_{kl}(p) \\ (k\&l=1,2,\cdots,N) \end{matrix} & \begin{matrix} -1 & & 0 \\ & -1 & \\ 0 & & \ddots \; -1 \end{matrix} & 0 \\
\hdashline
n_n'\text{ rows} & 0 & \multicolumn{2}{c|}{\begin{matrix} (k,n) \\ (k=1,2,\cdots,n_e)\ (n=1,2,\cdots,n_n') \end{matrix}} \\
\hdashline
n_m\text{ rows} & \begin{matrix} [k,m] \\ (k=1,2,\cdots,N) \\ (m=1,2,\cdots,n_m) \end{matrix} & \multicolumn{2}{c|}{0} \\
\hline
\end{array}
=0
$$

where N is the number of elements without voltage sources and n_{vs} is the number of elements with voltage sources, assuming the latter to be numbered last with the numbers $N+1$, $N+2$, ..., $n_e=N+n_{vs}$, and Y_{kl} is the admittance at the generalized frequency $p=\sigma+i\omega$ between the elements k and l, namely:

$$Y_{kl}=G_k+pC_k+\Gamma_k/p \quad \text{if } k=l, \quad \text{and} \quad Y_{kl}=\Gamma_{kl}/p \quad \text{if } k\neq l.$$

[Hint: Make use of the results of Ch. IV, §7 and §6, after substituting exponentially modulated sinusoids for all the time functions in the general equations (4, 5 & 6) of Ch. III, §2, and then make use of the known result on the existence of non-trivial solutions of homogeneous linear equations; finally make use of Laplace's development of a determinant to get rid of n_{vs} rows and n_{vs} columns corresponding to the coefficients of the unknowns I_{vs_k}.] (A simpler result will be given in §4 of Ch. VII.)

Problem 23. Show that the general complex equations of a sinusoidal current network (at the angular frequency ω) of general mixed series elements of the kind shown in Fig. 13 are the same as the eqs. 16, 17 & 18 of §1 (p. 125) but with: $Z_{kl}=R_k+i\omega L_k+1/(G_k+i\omega C_k)$ if $k=l$, and $Z_{kl}=i\omega L_{kl}$ if $k\neq l$, instead of eqs. (15).

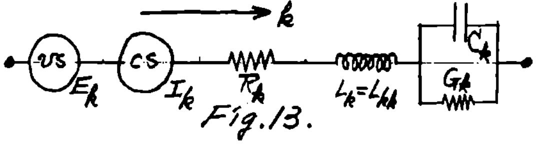


Fig. 13.

§3. NETWORKS OF GENERAL MIXED ELEMENTS IN THE SINUSOIDAL STATE.

Let us consider an arbitrary network of n_e general mixed series elements of the kind (whose transform is) shown in Fig. 1, (see next page) in which all the currents and voltages are sine functions of the same angular frequency $\omega>0$. Let an arbitrary reference direction be assigned to each of these elements and let us number them consecutively from 1 to n_e. Assume the network to have n_c components (separate connected parts) and n_n nodes, so that there are only $n'_n=n_n-n_c$ independent nodes which, after omitting exactly one node in each component, are numbered consecutively from 1 to n'_n. Assume that $n_m=n_e-n'_n$ independent oriented meshes are chosen in the network and that they are numbered consecutively from 1 to n_m.

Then we will have for the complex voltage drop in the current source of the element *k*, in the assigned reference direction:

$$V_{cs_k}=-D_k = Z'_k(I_k-I_{cs_k}) = Z'_kI_k - Z'_kI_{cs_k}, \quad (k=1,2,\dots,n_e) \qquad (1)$$

assuming Z'_k to be magnetically isolated; and for the complex current through the element *k*, we will have:

$$I_k = (G_k+i\omega C_k)V_{C_k}, \qquad (2)$$

so that the complex voltage drop in the "leaky" condenser shall be:

$$V_{C_k} = I_k/(G_k+i\omega C_k). \quad (k=1,2,\dots,n_e) \qquad (3)$$

We will then have for the total complex voltage drop in the general mixed series element *k*, in its assigned reference direction:

$$\begin{aligned} V_k &= V_{R_k} + V_{C_k} + V_{L_k} + V_{vs_k} + V_{cs_k} \\ &= R_kI_k + I_k/(G_k+i\omega C_k) + \sum_{l=1}^{n_e} i\omega L_{kl}I_l - E_k + Z'_k(I_k-I_{cs_k}) \\ &= \sum_{l=1}^{n_e} Z_{kl}I_l - E_k - Z'_kI_{cs_k}, \quad (k=1,2,\dots,n_e) \end{aligned} \qquad (4)$$

where we have put: $Z_{kl} = i\omega L_{kl}$ if $k\neq l$, and:

$$Z_{kk} = Z_k(\text{say}) = R_k + i\omega L_{kk} + 1/(G_k+i\omega C_k) + Z'_k. \qquad (5)$$

Besides the above equations we have, of course, Kirchhoff's Laws:

$$\sum_{k=1}^{n_e}(k,n)I_k = 0, \qquad (n=1,2,\dots,n'_n=n_n-n_c) \qquad (6)$$

$$\sum_{k=1}^{n_e}[k,m]V_k = 0, \qquad (m=1,2,\dots,n_m=n_e-n'_n). \qquad (7)$$

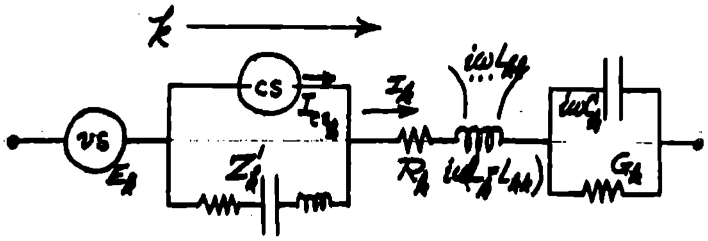


Fig. 1.

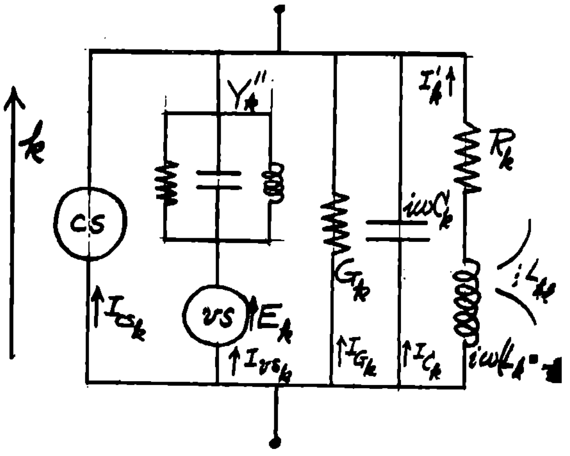


Fig. 2.

Let us next consider an arbitrary network of n_e general mixed parallel elements of the kind (whose transform is) shown in Fig. 2, in which all the currents and voltages are sine functions of the same angular frequency $\omega > 0$. Let n_n, n_c, $n'_n = n_n - n_c$, $n_m = n_e - n'_n$ have the same meaning as usual, and let all the elements and independent meshes be assigned reference directions arbitrarily, also as usual.

Then we will have for the complex current through the voltage source of the element k, in the assigned reference direction:

$$I_{vs_k} = Y''_k(V_k + E_k) = Y''_k V_k + Y''_k E_k, \qquad (k = 1, 2, \cdots, n_e) \tag{8}$$

where V_k is the total complex voltage drop across the element k, and Y''_k is assumed to be magnetically isolated. Also, for the complex current through the coil, we have (see Prob. 19, eq. (14), of § 2):

$$I_{L_k} = I'_k = \sum_{l=1}^{n_e} Y'_{kl} V_l, \qquad Y'_{kl} = \mathrm{cof}\,(R_k \delta_{kl} + i\omega L_{kl})/\det\,(R_k \delta_{kl} + i\omega L_{kl}), \tag{9}$$

where, as usual, $\delta_{kl} = 1$ if $k = l$ and $= 0$ if $k \neq l$.

Consequently, for the (total terminal) complex current entering the generic element k in the assigned reference direction, we have:

$$\begin{aligned} I_k &= I_{cs_k} + I_{vs_k} + I_{G_k} + I_{C_k} + I_{L_k} (= I'_k) \\ &= I_{cs_k} + Y''_k(V_k + E_k) + G_k V_k + i\omega C_k V_k + \sum_{l=1}^{n_e} Y'_{kl} V_l \\ &= \sum_{l=1}^{n_e} Y_{kl} V_l + I_{cs_k} + Y''_k E_k, \qquad (k = 1, 2, \ldots, n_e) \end{aligned} \tag{10}$$

where we have put: $Y_{kl} = Y'_{kl}$ for $k \neq l$, and (for $k = l$):

$$Y_{kk} = Y_k (\text{say}) = G_k + i\omega C_k + Y'_{kk} + Y''_k. \tag{11}$$

Besides the above equations we have Kirchhoff's complex current and voltage laws, of course, namely:

$$\sum_{k=1}^{n_e} (k,n) I_k = 0 \quad (n = 1, 2, \cdots, n'_n) \quad \& \quad \sum_{k=1}^{n_e} [k,m] V_k = 0. \quad (m = 1, 2, \cdots, n_m). \tag{12}$$

CHAPTER VI: THE MESH METHOD.

In Ch. V, §1 (p. 129) we have outlined a general procedure to solve the complex (transformed) equations of any network of general series elements in the sinusoidal state. In this chapter we will consider another method of solving these equations, known as the mesh method, which greatly simplifies matters. This method consists essentially in making the following substitution:

$$I_k = \sum_{m=1}^{n_m} [k,m]\, J_m \qquad (k = 1,2,\dots,n_e) \tag{1}$$

of the complex currents I_k *in terms of new (complex) quantities* J_m, *one for each mesh m (=1,2,...,*n_m*) of a complete and independent set of* n_m *meshes of the network, known as complex mesh (or circulating) currents (first introduced in their original form by J.Clerk Maxwell).*

The importance of the substitution (or change of variables) (1) is due to the theorem expressed by eq. (1) of Ch. II, §2, by means of which we can show that the mere form of the complex currents I_k *given by the substitution (1) is enough to guarantee that Kirchhoff's Current Law shall be automatically satisfied for all the nodes of the network, no matter what values the* J_m *may have. It shall then only be a question of finding the* J_m *so as to satisfy the other equations of the network.*

§1. THE CANONICAL EQUATIONS OF THE MESH METHOD.

Consider the general complex equations (16, 17 & 18 of Ch. V, §1) of an arbitrary network of n_e *general series elements in the sinusoidal state at the angular frequency* $\omega > 0$ *connected into* n_c *components with* n_n *nodes and* n_m *independent meshes, which we reproduce:*

$$V_k = \sum_{l=1}^{n_e} Z_{kl} I_l - E_k - D_k\,, \qquad (k = 1,2,\dots,n_e) \tag{2}$$

$$\sum_{k=1}^{n_e} (k,n) I_k = 0, \qquad (n = 1,2,\dots,n'_n = n_n - n_c) \tag{3}$$

$$\sum_{k=1}^{n_e} [k,m] V_m = 0, \qquad (m = 1,2,\dots,n_m = n_e - n'_n) \tag{4}$$

where the Z_{kl} *are the impedances of and between the elements of the network, at the angular frequency* $\omega (> 0)$, *given by eqs. (15), Ch. V, §1.*

Substituting (1) into the left-hand member of (3), we get:

$$\sum_{k=1}^{n_e} (k,n)\Big(\sum_{m=1}^{n_m} [k,m] J_m\Big) = \sum_{m=1}^{n_m} \Big\{\sum_{k=1}^{n_e} (k,n)[k,m]\Big\} J_m = \sum_{m=1}^{n_m} 0 \cdot J_m \equiv 0,$$

since all the sums in the parentheses of the second member are zero for all m and n, according to eq. (1) of Ch. II, §2. Thus Kirchhoff's complex current law (3) is identically satisfied for all the nodes of the network.

If we now substitute (2) into (4) we get:

$$\sum_{k=1}^{n_e}\sum_{l=1}^{n_e}[k,m]Z_{kl}I_l = \sum_{k=1}^{n_e}[k,m](E_k + D_k), \qquad (m = 1,2,\ldots,n_m)$$

which becomes, upon substituting the I_l as given by (1) after the summation index m is changed into another letter, r, say:

$$\sum_{r=1}^{n_m}\left(\sum_{k=1}^{n_e}\sum_{l=1}^{n_e}[k,m]Z_{kl}[l,r]\right)J_r = \sum_{k=1}^{n_e}[k,m](E_k + D_k). \qquad (m = 1,2,\ldots,n_m) \qquad (5)$$

Besides these equations, we have an equation (1) for each element k with a current source, for which the corresponding complex currents I_k are (assumed) known. Thus, if the network has $n_{cs}>0$ current sources which, without loss of generality, we may assume to be numbered first, consecutively from 1 to n_{cs} $(\leq n_m)$, then besides the n_m equations (5) with n_m unknown J_m and n_{cs} unknown D_k we have the following n_{cs} equations:

$$\sum_{m=1}^{n_m}[k,m]J_m = I_k. \qquad (k = 1,2,\ldots,n_{cs} \leq n_m) \qquad (6)$$

and, of course, the D_k in the eqs. (5) for $k = n_{cs}+1,\ n_{cs}+2,\ldots,n_e$ may be omitted, so that they can then be rewritten as follows:

$$\sum_{r=1}^{n_m}\left(\sum_{k=1}^{n_e}\sum_{l=1}^{n_e}[k,m]Z_{kl}[l,r]\right)J_r = \sum_{k=1}^{n_e}[k,m]E_k + \sum_{k=1}^{n_{cs}}[k,m]D_k. \qquad (m=1,2,\ldots,n_m) \qquad (7)$$

(When the network has no current source, i.e. $n_{cs}=0$, the complex equations of the network are the eqs. (5) with all the D_k omitted. In such a case the eqs. (6) are also to be omitted, of course.)

Equations (6) and (7), when $n_{cs}>0$, or equations (5) with the D_k omitted, when $n_{cs}=0$, form a complete and independent system of n_m+n_{cs} equations with this same number of unknowns, if the impedances Z_{kl} and the complex EMFs. E_k are considered as arbitrary (cf. note of Ch. III, §1, p. 64). These complex linear equations with complex coefficients of an arbitrary constant parameter (stationary) network of general series elements in the sinusoidal state are called the complex canonical mesh equations of the network.

§2. MESH IMPEDANCES.

The equations (5) and (7), of §1, can be given a simpler appearance by introducing the quantities $z_{mn} = z_{mn}(i\omega)$ defined as follows:

$$z_{mn} = \sum_{k=1}^{n_e}\sum_{l=1}^{n_e}[k,m]Z_{kl}[l,n]. \qquad (m \,\&\, n = 1,2,\ldots,n_m) \qquad (1)$$

Equations (5) §1, can then be written as follows:

$$\sum_{n=1}^{n_m} z_{mn} J_n = \sum_{k=1}^{n_e} [k,m](E_k + D_k) = F_m \text{ (say) } (m = 1,2,\ldots,n_m) \qquad (2)$$

and similarly with eqs. (7), of §1.

When $n_{cs}>0$, the complete and independent system of complex canonical mesh equations of a (stationary) sinusoidal current network of n_e general series elements with n_m independent meshes and n_{cs} ($\leq n_m$) current sources can then be given in the following final form:

$$\begin{aligned} \sum_{n=1}^{n_m} z_{mn} J_n &= \sum_{k=1}^{n_e} [k,m] E_k + \sum_{k=1}^{n_{cs}} [k,m] D_k \quad (m=1,2,\ldots,n_m) \\ \sum_{m=1}^{n_m} [k,m] J_m &= I_k \,; \qquad (k = 1,2,\ldots,n_{cs} \leq n_m) \end{aligned} \qquad (3)$$

and when $n_{cs}=0$, the complex canonical mesh equations are simply:

$$\sum_{n=1}^{n_m} z_{mn} J_n = \sum_{k=1}^{n_e} [k,m] E_k = \tilde{E}_m \text{(say)}, (m = 1,2,\ldots,n_m) \qquad (4)$$

The quantities z_{mn} introduced by eqs. (1) are called the mesh impedances of the network at the angular frequency ω (which is the angular frequency at which the element impedances Z_{kl} are computed). It should be noticed that they depend only on the passive parts (i.e. the parameters) of the network, on the way the elements are interconnected, and on the angular frequency, ω, of the sources. The quantities z_{mm} are called the self-impedances of the meshes m(=1,…,n_m) and the quantities z_{mn} ($m \neq n$) are called the mutual-impedances of, or between, the meshes m and n (=1,2,…,n_m). (z_{mm} is also called the mutual impedance of the mesh m with itself.)

If we denote the real part of z_{mn} by r_{mn} and the imaginary part of z_{mn} by x_{mn}, we shall have:

$$z_{mn} = r_{mn} + i\, x_{mn} \,. \qquad (m \,\&\, n = 1,2,\ldots,n_m) \qquad (5)$$

The quantity r_{mn} is called the mutual-resistance between the meshes m and n, and the quantity x_{mn} is called the mutual-reactance between these meshes. When m=n, these quantities are also called the self-resistance and self-reactance, respectively, of the mesh m.

Later we will see how these mesh quantities can be obtained directly by inspection of the network, in terms of its parameters, and also how they can be determined experimentally.

Problem 1. Show that if $Z_{kl} = Z_{lk}$ for all k and l (=1,2,…,n_e) then $z_{mn} = z_{nm}$ for all m and n (=1,2,…,n_m); thence infer that also $r_{mn} = r_{nm}$ and $x_{mn} = x_{nm}$ for all m and n.

Problem 2. By substituting the values of the element impedances Z_{kl} (k & $l = 1,2,\dots,n_e$) given by eqs. (15) of Ch. V, §1, into eqs. (1), show that:

$$z_{mn} = r_{mn} + i\omega l_{mn} + s_{mn}/i\omega\,, \quad (m \,\&\, n = 1,2,\dots,n_m) \quad (6$$

where we have put:

$$r_{mn} = \sum_{k=1}^{n_e} [k,m][k,n] R_k\,, \qquad s_{mn} = \sum_{k=1}^{n_e} [k,m][k,n] S_k\,,$$
$$l_{mn} = \sum_{k=1}^{n_e}\sum_{l=1}^{n_e} [k,m] L_{kl} [l,n]. \quad (m \,\&\, n = 1,2,\dots,n_m) \qquad (7)$$

Thence show that:

$$x_{mn} = \omega l_{mn} - s_{mn}/\omega\,. \qquad (m \,\&\, n = 1,2,\dots,n_m) \qquad (8)$$

The s_{mn} are called the (self-and mutual-) mesh elastances, and the l_{mn} are called the (self-and mutual-) mesh inductances, between the meshes m & n.

Problem 3. Show that in a network without mutual-inductances (i.e with all $Z_{kl} = 0$ if $k \neq l$) the mesh impedances are given by:

$$z_{mn} = \sum_{k=1}^{n_e} [k,m][k,n]\, Z_k\,, \qquad (m \,\&\, n = 1,2,\dots,n_m) \qquad (9)$$

and the mesh inductances are given by the following formula (while the r_{mn} & s_{mn} are still given by the similar first two eqs. 7):

$$l_{mn} = \sum_{k=1}^{n_e} [k,m][k,n]\, L_k\,. \qquad (m \,\&\, n = 1,2,\dots,n_m) \qquad (10)$$

Problem 4. Show that the general term $[k,m] Z_{kl} [l,n]$ of the double sum (1) defining z_{mn} shall have the same sign as Z_{kl} if and only if the mesh m traverses the element k in the same relative sense as the mesh n traverses the element l (i.e. both in the same directions, or both in the opposite directions, assigned to the elements k and l). The same holds for the general term $[k,m] L_{kl} [l,n]$ of the double sum defining l_{mn} by the third of the eqs. (7).

Problem 5. Let $\sum\limits_{k\in m}$ denote a summation over those elements k belonging to the mesh m (i.e. for those elements k for which $[k,m] \neq 0$). Then eqs. (1) can clearly be written as follows:

$$z_{mn} = \sum_{k\in m}\sum_{l\in n} [k,m] Z_{kl} [l,n] = \sum_{\substack{k\in m\\ k\in n}} [k,m][k,n] Z_k + \sum_{\substack{k\in m\\ k\neq l}}\sum_{l\in n} [k,m] Z_{kl} [l,n]\,; \qquad (11)$$

and eqs. (7) can be written thus:

$$r_{mn} = \sum_{\substack{k\in m\\ k\in n}} [k,m][k,n] R_k\,, \qquad s_{mn} = \sum_{\substack{k\in m\\ k\in n}} [k,m][k,n] S_k\,,$$
$$l_{mn} = \sum_{k\in m}\sum_{l\in n} [k,m] L_{kl} [l,n] = \sum_{\substack{k\in m\\ k\in n}} [k,m][k,n] L_k + \sum_{\substack{k\in m\\ k\neq l}}\sum_{l\in n} [k,m] L_{kl} [l,n]. \qquad (12)$$

From the first of the eqs. (12) it can be inferred that the mutual-

resistance between the meshes m and n is an algebraic sum of all the resistances common to the two meshes (and no others), each resistance being taken with a positive sign if the two meshes (m & n) traverse it in the same direction, and with a negative sign otherwise. The same can be said about the mutual elastance, s_{mn}, between the meshes m and n given by the second of eqs. (12), since it has the same form as the first of eqs. (12). And the same can be said about the first term in the last member of eq. (11), and of the last of the eqs. (12), because they have the same form; and also of eqs. (9) & (10).

Problem 6. From the preceding problem (5) show that in the particular case of m=n, we have for the self-resistance, self-elastance, and self-inductance, of the mesh m, respectively:

$$r_{mm} = \sum_{k\in m} R_k , \qquad s_{mm} = \sum_{k\in m} S_k , \qquad (m = 1,2,\ldots,n_m)$$
$$l_{mm} = \sum_{k\in m} L_k + 2\sum_{k\in m}\sum_{l(>k)\in m} [k,m][l,m]\, L_{kl}. \qquad (13)$$

From the first (two) of these equations we see that the self-resistance (-elastance) of the generic mesh m is simply the arithmetical sum of all the resistances (elastances) in that mesh. And from the last of these equations, it can be seen that the self-inductance of the generic mesh m is equal to the arithmetical sum of all the self-inductances (of coils) in that mesh plus twice the algebraic sum of L_{kl} all the mutual-inductances between distinct coils of that mesh, each taken with the sign of [k,m][l,m].

Problem 7. From eqs. (7), or eqs. (12), show that $r_{mn} = r_{nm}$, $s_{mn} = s_{nm}$, and if $L_{kl} = L_{lk}$ for all k & l = 1,2,...,n_e, that $l_{mn} = l_{nm}$, for all m & n = 1,2,...,n_m.

Problem 8. Show that the mesh-impedances z_{mn} do not depend on the reference directions assigned to the elements. They depend only on the reference directions assigned to the meshes, in a given network. [HINT: Use eqs. (1) to show that changing the reference direction assigned to any one element does not affect any one of the z_{mn}, if the reference directions assigned to the meshes are unaltered.]

We will now examine eqs. (1) for the mesh impedances in more detail. In the first place we observe that for any given meshes m and n (not necessarily distinct), only those terms of the double sum appear for which $[k,m]\neq 0$, $[l,n]\neq 0$, and $Z_{kl}\neq 0$. This means that only those values of k which belong to elements of mesh m, and only those values of l which belong to elements of mesh n, need be considered. Denoting by $\sum_{k\in m}$ a summation over all values of k belonging to elements of the mesh m (as in Problem 5), eqs. (1) can be written as follows (for m & $n = 1,2,\dots,n_m$):

$$z_{mn} = \sum_{k\in m}\sum_{l\in n}[k,m]\,Z_{kl}\,[l,n] = \sum_{k\in m}[k,m]\Big[\sum_{l\in n}Z_{kl}\,[l,n]\Big]. \quad (14)$$

The practical way to evaluate this double sum, for any given meshes m and n, is as follows. An element k of the mesh m is chosen to begin with. Then we run through all the elements l of the mesh n (beginning with any one element of the mesh n) and compute all the terms $[k,m]Z_{kl}[l,n]$ with a $Z_{kl} \neq 0$. (Each one of these terms shall be $\pm Z_{kl}$, with a + sign if the elements k and l are traversed relatively in the same senses by the corresponding meshes m and n, respectively, and with a - sign if they are traversed relatively in opposite senses, as is shown in Fig. 1. This will be called the rule of signs for the impedances in the mesh method.) Then we pass on to the next element k' (say) of the mesh m (e.g. in the sense the mesh m is oriented) and again we run through all the elements l of the mesh n, computing, as we go along, all the terms $[k',m]Z_{k'l}[l,n]$ with a $Z_{k'l} \neq 0$; etc. We continue in this way until we have gone through all the elements k of the mesh m, each time (for each element k of the mesh m) going through all the elements l of the mesh n. Finally we add up all the terms so computed. (This is very much like adding up the terms in a rectangular array of numbers. First we choose a row and then we add up all the terms on this row. Then we choose another row and proceed as before. Finally we add up the results of the various rows.) In this way we obtain the impedance z_{mn} between the meshes m and n.

$[k,m]Z_{kl}[l,n] = +Z_{kl}$ in (a) and (b). $[k,m]Z_{kl}[l,n] = -Z_{kl}$ in (c) and (d).

Fig. 1. The rule of signs for the mesh method.

Example 1. The self-impedance z (say) of the mesh formed by the elements 1,4,5,6,3 of the network shown in Fig. 7, Ch. V, §1, is:

$$z = Z_{11} + Z_{14} + Z_{44} + Z_{41} + Z_{55} + Z_{66} + Z_{63} + Z_{33} + Z_{36}.$$

Problem 9. Compute the self-impedance of the mesh formed by the elements 1,2,3 of the same network considered in the preceding example 1; also compute the self-impedance of the mesh $(\bar{2},4,5,6)$. (See Ex. 6.)

Example 2. Consider the transformed network of that shown in Fig. 2, Ch. III, §1 (p.65). Assume now that the coils of the elements 7, 8, 9, 10 are coupled magnetically. Then the self-impedance $\underline{z}$ (say) of the mesh formed by the elements $7,10,\overline{8},\overline{9}$ is:

$$z = Z_7+Z_{7,10}-Z_{78}-Z_{79}+Z_{10,7}+Z_{10}-Z_{10,8}-Z_{10,9}-Z_{87}-Z_{8,10}+Z_8+Z_{89}+$$
$$-Z_{97}-Z_{9,10}+Z_{98} = Z_7+Z_8+Z_9+Z_{10}+2(Z_{7,10}-Z_{78}-Z_{79}-Z_{8,10}-Z_{9,10}+Z_{89}).$$

Problem 10. In the network of the preceding example 2, compute the self-impedance of the mesh (3,12,8,11) assuming that all the elements of this mesh have coils and that they are all coupled magnetically. (See Ex. 7.)

Example 3. Let us compute the mutual-impedance z_M (say) between the meshes (1,4,5,6,3) and $(\overline{1},\overline{3},\overline{2})$, oriented in the orders given, of the network considered in the example 1 (Fig. 7, Ch. V, §1, p. 123). It is:

$$z_M = -Z_1-Z_{12}-Z_{41}-Z_{42}-Z_{63}-Z_3.$$

If the second mesh had been taken as (1,2,3) then we would have obtained the negative of the above result.

Problem 11. In the same network (Fig. 7, Ch. V, §1) compute the mutual-impedance between the meshes (1,2,3) and $(2,\overline{6},\overline{5},\overline{4})$. (See Ex. 8.)

Example 4. In the network considered in the example 2, the mutual-impedance z_M (say) between the meshes $(\overline{7},9,8,\overline{10})$ and $(3,6,\overline{8},\overline{9})$, assuming (again) that, besides the non-zero mutual-inductances already shown in the network, there are also non-zero mutual-inductances between each pair of the elements 7, 8, 9 & 10, is:

$$z_M = -Z_{73}+Z_{78}+Z_{79}-Z_{98}-Z_9-Z_8-Z_{89}-Z_{10,6}+Z_{10,8}+Z_{10,9}.$$

Problem 12. Evaluate the mutual-impedance between the meshes $(\overline{6},\overline{7},\overline{11})$ and (3,10,11) of the same network considered in example 2, under the same hypotheses. (See Ex. 9.)

Problem 13. Evaluate the self-impedances of the meshes $(1,2,\overline{4})$, $(1,2,\overline{6},\overline{5})$, $(3,2,\overline{6})$ in the network shown in Fig. 9, Ch. V, §1, p. 126. Also compute the mutual-impedances between the following pairs of meshes in that network: $(1,2,\overline{4})$ & $(3,2,\overline{6})$, $(1,\overline{3},\overline{5})$ & $(2,\overline{6},3)$, $(1,2,\overline{4})$ & $(1,2,\overline{6},\overline{5})$, $(2,\overline{6},3)$ & $(7,\overline{8},\overline{9})$. (See Ex. 10.)

Problem 14. Show that if the reference direction of exactly one mesh $\underline{m}$ of a network is changed, then the self-impedance of this mesh $\underline{m}$ does not change, while all the mutual-impedances of this mesh $\underline{m}$ with the other meshes of the network change their sign (only). Use this result to show that the canonical mesh equations of a network remain (essentially) unaltered under a change of mesh orientations.

The mesh impedances z_{mn} obtained in terms of the element impedances Z_{kl} may be expressed in more detail in terms of the resistances, elastances, and (self- and mutual-) inductances, of the network by replacing (in accordance with the eqs. 15 of Ch. V, §1) each self-impedance $Z_k = Z_{kk}$ by $R_k + i\omega L_k + S_k/i\omega$ and each mutual-impedance Z_{kl} $(k \neq l)$ by $i\omega L_{kl}$. We thus see that any resistance or elastance appearing in a mesh impedance z_{mn} can only be due to a self-impedance of an element on both meshes $\underline{m}$ and $\underline{n}$; and conversely, any resistance or elastance on both meshes $\underline{m}$ and $\underline{n}$ must necessarily appear in z_{mn}. All other terms in z_{mn} shall be of the form $i\omega L_{kl}$.

It may be well to mention that just as the sum of the terms of a rectangular array can be obtained in various other ways besides that of adding by rows (e.g. by adding by columns, by diagonals, by rectangles, etc.) so can that of the double summation in eqs. (1) for the mesh impedance z_{mn}. Thus (for example) we may first find the sum of all the terms with $k=l$, corresponding to the terms on the main diagonal, and then we may add the sum of all the terms with $k \neq l$, corresponding to the terms off the main diagonal; the latter, in turn, may be split into: those terms corresponding to distinct elements $\underline{k}$ & $\underline{l}$ belonging to both the meshes $\underline{m}$ and $\underline{n}$, and those terms corresponding to distinct elements $\underline{k}$ & $\underline{l}$ not (both) belonging to both the meshes $\underline{m}$ and $\underline{n}$. It is clear that in this way all the terms of the double summation (1) are split into mutually exclusive classes of terms, so that each term is included and no term is duplicated.

All the terms with $k=l$ correspond to elements common to both the meshes $\underline{m}$ & $\underline{n}$, and they are all of the form $[k,m]Z_{kk}[k,n] = \pm Z_{kk} = \pm Z_k$, which are precisely the self-impedances of those elements, each taken with a sign according to the rule of signs exhibited in Fig. 1. In particular, when $m=n$ (i.e. when the meshes $\underline{m}$ & $\underline{n}$ are the same mesh) then $[k,m][k,n] = [k,m]^2 = 1$, so that the self-impedance Z_k of every element $\underline{k}$ on the mesh $\underline{m}$ appears with a + sign in the self-impedance z_{mm} of the mesh $\underline{m}$.

All the terms with $k \neq l$ corresponding to pairs of distinct elements (both) common to both meshes $\underline{m}$ & $\underline{n}$ always appear in pairs: $[k,m]Z_{kl}[l,n] + [l,m]Z_{lk}[k,n]$; because the element $\underline{k}$ is on both meshes $\underline{m}$ & $\underline{n}$, and so is element $\underline{l}$, and so $[k,m] \neq 0$, $[k,n] \neq 0$, $[l,m] \neq 0$, $[l,n] \neq 0$. Now since $L_{kl} = L_{lk}$ (for constant parameter networks, as is (always assumed) true in alternating current networks), we have $Z_{kl} = Z_{lk}$. Therefore the above pair of terms can be written $([k,m][l,n] + [l,m][k,n])Z_{kl}$; but the quantity in parenthesis can only be, in absolute value, either 2 or 0, according to whether the elements $\underline{k}$ & $\underline{l}$ are in the situation (a) or (b), respectively, shown in Fig. 2. (It can be shown that any case can be reduced to exactly one of these.) Consequently the mutual impedances $\pm Z_{kl}$ between distinct elements which are (both) common to both meshes $\underline{m}$ & $\underline{n}$ either appear twice or not at all. In particular, when $m=n$, we can only have the situation (a) of Fig. 2, so that in a self-impedance z_{mm} of a mesh $\underline{m}$, all non-zero mutual-impedances Z_{kl} between distinct elements of the mesh $\underline{m}$ appear twice.

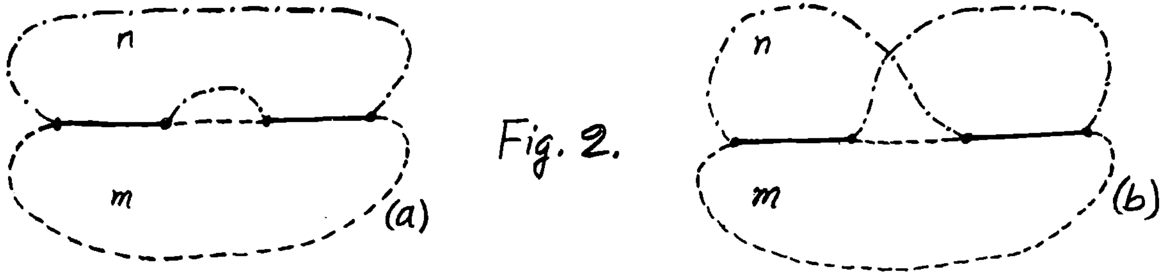


Fig. 2.

All the terms with $k \neq l$ corresponding to pairs of distinct elements, one on each mesh but not (both) common to both meshes $\underline{m}$ & $\underline{n}$, are all of the form $[k,m] Z_{kl} [l,n] = \pm Z_{kl}$ and can only appear singly (not in pairs) and this, in fact, only when $m \neq n$, each with a sign given by the rule of signs. When $m=n$, these terms are not present.

We can summarize all the above into the following practical rule for the computation (by inspection) of the mesh impedances of a network. First we shall consider the case $m=n$ of a self-impedance of a mesh, and then the case $m \neq n$ of a mutual-impedance between meshes.

1º). The self-impedance z_{mm} of mesh $\underline{m}$ is the sum of all the self-impedances of all the elements of this mesh (all taken with a + sign) plus twice the (algebraic) sum of all the (non-zero) mutual-impedances between pairs of distinct elements of this mesh (each taken with a sign given by the rule of signs exhibited in Fig. 1).

By substituting the impedances of the elements according to eqs. (15), Ch. V, §1, we can state this rule in more detail as follows: z_{mm} is the sum of all the resistances in mesh $\underline{m}$ (all taken with a + sign), plus $1/i\omega$ multiplied by the sum of all the elastances in mesh $\underline{m}$ (all taken with a + sign), plus $i\omega$ multiplied by the sum of all the self-inductances (of coils) in mesh $\underline{m}$ (all taken with a + sign) in addition to twice the (algebraic) sum of all the mutual-inductances between pairs of distinct coils in mesh $\underline{m}$ (each of these taken with a sign given by the rule of signs given above and exhibited in Fig. 1).

29). The mutual-impedance z_{mn} between two distinct meshes m & n is the (algebraic) sum of all the self-impedances of all the elements common to both meshes (each one taken with a + sign if the two meshes m & n traverse the element in the same direction and with a - sign if they do so in opposite directions, irrespective of the reference direction assigned to the element) plus twice the (algebraic) sum of all the non-zero mutual-impedances between pairs of distinct elements (both) common to both meshes m & n (if and only if they are in the situation (a) of Fig. 2 ; each taken with a sign given by the rule of signs exhibited in Fig. 1) plus the (algebraic) sum of all the non-zero mutual-impedances between pairs of elements, one on each mesh but not (both) common to both the meshes m & n (each taken with a sign in accordance with the rule of signs, Fig. 1).

By substituting the impedances of the elements according to eqs. (15), Ch. V, §1, we can state this rule in more detail as follows: z_{mn} ($m \neq n$) is the (algebraic) sum of all the resistances common to both meshes m & n plus $1/i\omega$ multiplied by the (algebraic) sum of all the elastances common to the two meshes plus $i\omega$ multiplied by the (algebraic) sum of all the self-inductances on both meshes (each of the foregoing resistances, elastances, and self-inductances, being taken with a + sign if the two meshes m & n traverse it in the same direction and with a - sign if they do so in opposite directions, no matter how the elements are oriented) plus $i\omega$ multiplied by twice the (algebraic) sum of all the non-zero mutual-inductances between pairs of distinct coils (both) common to both meshes (if and only if they are in the case (a) of Fig. 2; each taken with a sign given by the rule of signs exhibited in Fig. 1) plus $i\omega$ multiplied by the (algebraic) sum of all the non-zero mutual-inductances between pairs of coils, one on each mesh but not (both) common to both meshes m & n (each taken with a sign given by the rule of signs exhibited in Fig. 1).

When polarity marks for the mutual-inductances between pairs of coils are given, the final signs with which their absolute values ~~they~~ shall appear in the (self- and mutual-) mesh impedances shall depend only on the reference directions assigned to the meshes; because, if $k \neq l$ and element k is on mesh m while element l is on mesh n, no matter how these elements are oriented the incidence numbers $[k,m]$ and $[l,n]$ in the term $[k,m] Z_{kl} [l,n] = i\omega [k,m] L_{kl} [l,n]$ of z_{mn} will automatically take care of referring everything back to the reference directions assigned to the meshes. In particular, when m=n, the absolute value of a non-zero mutual-inductance between *two distinct coils of the mesh* m *shall appear in* z_{mm} *with a + sign if the arrow assigned to the mesh is directed towards the two corresponding polarity marks or if it is directed away from both of them, and it shall appear with a - sign otherwise (see Fig. 3). In the case of two distinct meshes* m *&* n*, the absolute value of a mutual-inductance* L_{kl} *between two distinct coils* k *&* l *(coil* k *on mesh* m *and coil* l *on mesh* n*) shall appear in* z_{mn} *with a + sign if the two meshes* m *&* n *are oriented relatively the same with respect to the polarity marks on the coils* k *&* l *(i.e. if the meshes traverse the corresponding elements both towards or both away from their polarity marks) and with a - sign otherwise (see Fig. 4).*

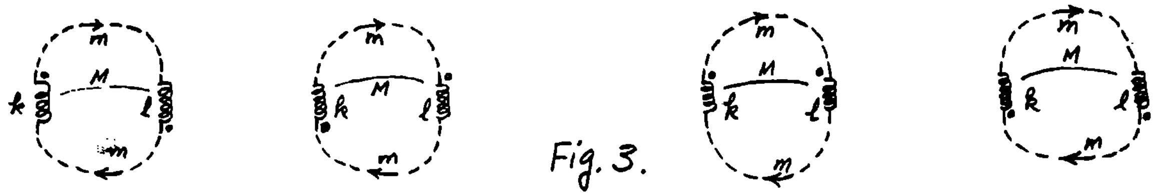


Fig. 3.

$M=|L_{kl}|$ appears with a + sign in z_{mm}.

$M=|L_{kl}|$ appears with a - sign in z_{mm}.

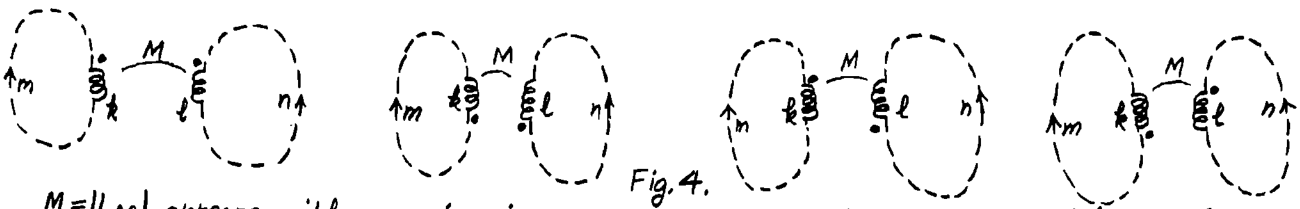


Fig. 4.

$M=|L_{kl}|$ appears with a + sign in z_{mn}.

$M=|L_{kl}|$ appears with a - sign in z_{mn}.

Example 5. The self-impedance z of the mesh formed by the elements 1,4,5,6,3 of the network shown in Fig. 7, Ch. V, §1 (p.123) is:

$$z=(R_1+R_4+R_5+R_6+R_3)+S_5/i\omega+i\omega(L_1+L_4+L_5+L_6+L_3+2[L_{14}+L_{36}]).$$

This can be checked with the result given in example 1 (above).

Example 6. The self-impedances of the meshes (1,2,3) & ($\bar{2}$,4,5,6) of the same network considered in the preceding example 5, are (resp.):

$$z=(R_1+R_2+R_3)+S_2/i\omega+i\omega[L_1+L_2+L_3+2L_{12}],$$

$$z'=(R_2+R_4+R_5+R_6)+(S_2+S_5)/i\omega+i\omega[L_2+L_4+L_5+L_6+2(-L_{24})].$$

Problem 15. Compute the self-impedance of the mesh (7,10,$\bar{8}$,$\bar{9}$) of the network (Fig. 2, Ch. III, §1, p. 65) considered in the example 2 given above, assuming that the coils of the elements 7, 8, 9, 10, are magnetically coupled, in terms of R's, S's, and L's. (Check with Ex. 2.)

Example 7. The self-impedance asked for in Problem 10 is, in detail, the following:

$$z=(R_3+R_{12}+R_8+R_{11})+\frac{1}{i\omega}(S_3+S_{12}+S_8+S_{11})+i\omega[L_3+L_{12}+L_8+L_{11}+2(L_{3,12}+L_{38}+L_{3,11}+L_{12,8}+L_{12,11}+L_{8,11})].$$

Problem 16. Compute the mutual-impedance between the meshes (1,4,5,6,3) and (1,2,3) of the network considered in Example 5. (See Ex. 3.)

Example 8. In the same network (Fig. 7, Ch. V, §1, p. 123), consider the meshes (1,2,3) & $(2,\bar{6},\bar{5},\bar{4})$. The mutual-impedance is:

$$z_M = R_2 + S_2/i\omega + i\omega(L_2 - L_{14} + L_{12} - L_{24} - L_{36}).$$

Problem 17. Compute in detail the mutual-impedance between the meshes $(\bar{7},9,8,\overline{10})$ & $(3,6,\bar{8},\bar{9})$ of the network considered in example 2 (Fig. 2, Ch. III, §1, p. 65), assuming that there are also non-zero mutual-inductances between each pair of the elements 7, 8, 9, 10, besides the non-zero mutual-inductances already indicated in the figure. (See Ex. 4.)

Example 9. In the same network (Fig. 2, Ch. III, §1), the mutual-impedance between the meshes $(\bar{6},\bar{7},\overline{11})$ and (3,10,11) is:

$$z_M = -R_{11} - S_{11}/i\omega + i\omega(-L_{6,10} - L_{7,3} - L_{7,10}).$$

Example 10. In the network shown in Fig. 9, Ch. V, §1 (p. 126), the self-impedances of the meshes $(1,2,\bar{4})$, $(1,2,\bar{6},\bar{5})$, $(3,2,\bar{6})$, and the mutual-impedances between the following pairs of meshes: $(1,2,\bar{4})$ & $(3,2,\bar{6})$, $(1,\bar{3},\bar{5})$ & $(2,\bar{6},3)$, $(1,2,\bar{4})$ & $(1,2,\bar{6},\bar{5})$, $(2,\bar{6},3)$ & $(7,\bar{8},\bar{9})$ are (respectively):

$$z = (R_1+R_2+R_4)+(S_1+S_4)/i\omega + i\omega(L_1+L_2+L_4+2L_{12}), \quad (\text{for mesh } 1,2,\bar{4})$$

$$z' = (R_1+R_2+R_6+R_5)+(S_1+S_6+S_5)/i\omega + i\omega(L_1+L_2+L_6+L_5+2L_{12}), \quad (\text{for mesh } 1,2,\bar{6},\bar{5})$$

$$z'' = (R_3+R_2+R_6)+(S_3+S_6)/i\omega + i\omega(L_3+L_2+L_6+2L_{32}), \quad (\text{for mesh } 3,2,\bar{6})$$

$$z_M = R_2 + i\omega L_2 + i\omega(L_{13}+L_{12}+L_{23}+L_{46}), \quad (\text{for meshes } 1,2,\bar{4} \ \& \ 3,2,\bar{6})$$

$$z'_M = -R_3 - S_3/i\omega - i\omega L_3 + i\omega(L_{12}+L_{13}-L_{32}), \quad (\text{for meshes } 1,\bar{3},\bar{5} \ \& \ 2,\bar{6},3)$$

$$z''_M = R_1+R_2+S_1/i\omega + i\omega(L_1+L_2+2L_{12}) + i\omega L_{46}, \quad (\text{for meshes } 1,2,\bar{4} \ \& \ 1,2,\bar{6},\bar{5})$$

$$z'''_M = i\omega(-L_{67}) = -i\omega L_{67}. \quad (\text{for meshes } 2,\bar{6},3, \ 7,\bar{8},9)$$

Example 11. Consider the network shown in Fig. 10 (g) of Ch. V, §1 (p. 128). Assume that the M's are the absolute values of the mutual-inductances. The self-impedance of the mesh (1,2,3) is:

$$z = R_1+R_3+S_2/i\omega + i\omega(L_1+L_2+L_3+2[-M+M'])$$

and the mutual-impedance between this mesh and that formed by the element 4 (oriented as shown) is:

$$z' = i\omega(M'' - M''').$$

Problem 18. Compute the self- and mutual-impedances of the meshes 1 & 2 of the network shown in Fig. 10 (j) of Ch. V, §1 (p. 128) in terms of the resistances, elastances and the absolute values of the inductances.

§3. THE MESH CURRENTS AND VOLTAGES AND THEIR INTERPRETATION.

It is easy to give an interpretation of the complex mesh (circulating) currents J_m $(m=1,2,\dots,n_m)$ introduced by eqs. (1) of the introduction to this chapter. For this purpose, assume that all the J_m are zero except $J_{m'}$ (say); then according to those equations we have for all $k=1,2,\dots,n_e$ that $I_k=[k,m']J_{m'}$. From this we see that all the complex currents I_k are zero except those in the elements of the mesh m'; because $[k,m']=0$ for the elements k not on the mesh m'. We also see that the complex currents through the elements of the mesh m', in the direction assigned to the mesh, are all the same; because, if the element k is directed in the sense of the mesh then $[k,m']=1$ and so $I_k=J_{m'}$, and if the element k is directed in a sense opposite to that of the mesh then $[k,m']=-1$ and so the complex current through the element k in the reference direction assigned to the mesh m' is $-I_k=-[k,m']J_{m'}=J_{m'}$. Consequently $J_{m'}$ is a complex current flowing in a closed path around the mesh m'; for this reason it is also called a *complex circulating current*.

If the mesh m traverses the element k in the reference direction assigned to it then $[k,m]=1$, and the contribution of J_m to the complex current I_k through the element k shall be $J_m=[k,m]J_m$. If the mesh m traverses the element k in the opposite direction then $[k,m]=-1$, and the contribution of J_m to I_k shall be $-J_m=[k,m]J_m$. Moreover, if the mesh m does not traverse the element k then $[k,m]=0$, so that the contribution, which in this case is 0, is also given by $[k,m]J_m$. Hence in every case, $[k,m]J_m$ is the contribution of the complex mesh current J_m to the complex current I_k through the element k; and so the equation:

$$I_k=\sum_{m=1}^{n_m}[k,m]J_m \qquad (k=1,2,\dots,n_e) \qquad (1)$$

then expresses that the complex current through the element k (in the reference direction assigned to it) is the (algebraic) sum of all the complex circulating currents passing through it, each one passing in the reference direction assigned to the element taken with a + sign and the ones passing in the opposite sense taken with a - sign. (In the case of an element with a current source, the corresponding eq. 1 tells us that this sum adds up to the complex value of the current source, as it should be.) Based on this, all the eqs. (1) can be written down by inspection of any given network, as soon as all the reference directions are assigned to the elements and to all the meshes (of a complete and independent set of meshes) of the network.

It is also easy to give an interpretation of the right-hand members of eqs. (2), (4), and of the first of eqs. (3), of §2. For E_k and D_k are the complex voltage rises in the voltage and current sources of the element $\underline{k}$ (in the assigned reference direction). Therefore, $[k,m]\cdot E_k$ and $[k,m]\, D_k$ are the complex voltage rises in these sources, but now referred to the direction assigned to the mesh $\underline{m}$, if the element $\underline{k}$ is on the mesh $\underline{m}$. (Thus multiplication of a voltage rise in the reference direction assigned to an element $\underline{k}$ by the incidence number of the element $\underline{k}$ with a mesh $\underline{m}$, gives the voltage rise in the reference direction assigned to the mesh.) Hence the right-hand members of eqs. (2), §2, are the (algebraic) sums of all the complex voltage rises in all the (current and voltage) sources in the meshes $\underline{m}$, in the reference directions assigned to the meshes. Accordingly, the right-hand members of eqs. (2), (3), & (4), of §2, can be written down immediately by inspection for all the chosen meshes $m = 1,2,\ldots,n_m$ of a complete and independent set of meshes of any given network.

Moreover, it is quite an easy matter to write down the left-hand members of these equations systematically. One needs only to formally write down expressions (sums) such as: $z_{m1}J_1 + z_{m2}J_2 + \ldots + z_{mn_m}J_{n_m}$, with as many terms n_m as there are independent meshes in the network, one for each independent mesh $\underline{m}$. Furthermore, the complex coefficients z_{mn} (m & $n = 1,2,\ldots,n_m$) in these expressions may readily be obtained by inspection of the given network, by the methods and rules given in the preceding section (§2). Thus all the complex canonical equations of any given (stationary) network can be completely established directly by inspection of the network.

Essentially, eqs. (2) (and the first of eqs. 3, and eqs. 4) of §2, express Kirchhoff's complex voltage law for the meshes $\underline{m} = 1,2,\ldots,n_m$. And since the right-hand members of these equations are the (algebraic) sums of all the complex voltage rises in the sources of the meshes $\underline{m}$, the left-hand members must be the (algebraic) sums of all the complex voltage drops in all the other (i.e. in the passive parts of the) elements of these meshes, all taken in the reference directions assigned to the~~m~~ meshes. The contribution of the complex mesh (circulating) current J_n ($n = 1,2,\ldots,n_m$) to the (algebraic) sum of complex voltage drops in (the passive parts of) the typical mesh $\underline{m}$ is then $z_{mn}J_n$. Consequently, z_{mn} may be interpreted as the (algebraic) sum of the complex voltage drops in the passive parts of the elements of the mesh $\underline{m}$ (in the reference direction assigned to the mesh m) due to a unit complex current $1\angle 0^\circ$ circulating around the mesh n (in the reference direction assigned to the mesh n). In particular (when $m=n$), z_{mm} is the (algebraic) sum of the complex voltage drops in the passive parts of the elements of the mesh m due to a unit complex current circulating around it. These remarks enable us to obtain experimentally all the mesh impedances z_{mn} ($m \& n = 1,2,\ldots,n_m$) (see problem 1, below). They (the remarks) also give us a method to obtain them (the mesh impedances) by inspection. In fact, this is the method found (and used) in the older literature. It consists in imagining a unit complex current circulating around the mesh n and then determining by inspection (or experimentally) the complex voltage drops it produces in the passive parts of the elements of the mesh m (by actually passing through them or by virtue of the mutual-inductances), the (algebraic) sum of which is z_{mn}. Of course, now that we have eqs. (1), §2, to determine the mesh impedances, the older method is not necessary. Actually, these equations (or the equivalent rules given in §2) will give us the mesh impedances faster, easier, and with much less probability of making an error; and with them we can easily obtain the mesh impedances from the table (matrix) of the element-mesh incidence numbers, or from a conventional graph in which identification of terminals is used.

Problem 1. Assume we are given an arbitrary a-c network. Suppose we "short-circuit" all the sources, but otherwise leave the rest of the (basic) elements as they were. This, of course, shall not affect the mesh impedances; because they do not depend on the sources (except for their frequencies). Now open (in any way) all the meshes except mesh n, into which a sinusoidal (voltage or current) source is inserted. Let $\sqrt{2}A \sin \omega t$ (with $A>0$) be the complex current resulting in this mesh n, so that the complex current in it is $A\angle 0^\circ$. Let $B\angle\beta$ (with $B>0$) denote the (algebraic) sum of all the complex voltage rises across the gaps (i.e. the openings produced) in the mesh m. (All these reckoned in the reference directions assigned to the meshes.) Show that $z_{mn} = z_{mn}(i\omega) = (B/A)\angle\beta$. By measuring the quantities A, B, and β, the mesh impedance z_{mn} between any two meshes m & n can be determined experimentally (and when $m=n$ we obtain z_{mm}, of course).

The terms $z_{mn}J_n$ in the left-hand members of eqs. (2, 3, & 4), §2, will be called zJ-drops. The term $z_{mm}J_m$ will be called the self-zJ-drop around (or in) the mesh m due to its own complex mesh current J_m, and $z_{mn}J_n$ ($m \neq n$) will be called the mutual-zJ-drop around (or in) the *mesh $\underline{m}$ due to the complex mesh current J_n around (in, or of) the mesh $\underline{n}$ (all referred to the orientations assigned to the meshes). We can then say that the (algebraic) sum of all the complex voltage drops in the passive parts of the elements of a mesh is equal to the sum of all the self and mutual zJ-drops around that mesh; and this, of course is also equal to the (algebraic) sum of all the complex voltage rises in the sources of that mesh, as is indicated by the right-hand members of eqs. (2 & 4), and of the first of eqs. (3), §2.*

Example 1. The complex canonical mesh equations of the network shown in Fig. 7, Ch. V, §1, for which $n_m=3$, are (using the same complete and independent set of meshes: 123, 2456, & 7, used in the example 4 of Ch. V, §1, p. 123):

$$z_{11}J_1 + z_{12}J_2 + z_{13}J_3 = E_1 \text{ (known)}$$
$$z_{21}J_1 + z_{22}J_2 + z_{23}J_3 = 0$$
$$z_{31}J_1 + z_{32}J_2 + z_{33}J_3 = D_7$$
$$J_3 = I_7 \text{ (known)}$$

where:

$$z_{11}=R_1+R_2+R_3+S_2/i\omega+i\omega(L_1+L_2+L_3+2L_{12})$$
$$z_{12}=-R_2-S_2/i\omega-i\omega L_2+i\omega(L_{14}-L_{12}+L_{24}+L_{36}) = z_{21}$$
$$z_{13} = z_{31} = 0$$
$$z_{22}=R_2+R_4+R_5+R_6+(S_2+S_5)/i\omega+i\omega(L_2+L_4+L_5+L_6-2L_{24})$$
$$z_{23} = z_{32} = i\omega L_{57}$$
$$z_{33} = R_7 + S_7/i\omega + i\omega L_7.$$

These results can be checked with the results of Prob. 3, Ch. V, §1.

Example 2. Consider the network of example 5, Ch. V, §1 (p. 126) and let us take the same complete independent set of five meshes used in that example. The complex canonical mesh equations are:

$$z_{11}J_1 + z_{12}J_2 + z_{13}J_3 + z_{14}J_4 + z_{15}J_5 = E_1 - E_4 - D_4$$
$$z_{21}J_1 + z_{22}J_2 + z_{23}J_3 + z_{24}J_4 + z_{25}J_5 = E_5 - E_4 + D_5 + D_6 - D_4$$
$$z_{31}J_1 + z_{32}J_2 + z_{33}J_3 + z_{34}J_4 + z_{35}J_5 = E_1 - E_3 - E_5 - D_5$$
$$z_{41}J_1 + z_{42}J_2 + z_{43}J_3 + z_{44}J_4 + z_{45}J_5 = E_{10} + E_{11}$$
$$z_{51}J_1 + z_{52}J_2 + z_{53}J_3 + z_{54}J_4 + z_{55}J_5 = -D_8$$

where:

$$z_{11} = R_1+R_2+R_4+(S_1+S_4)/i\omega+i\omega(L_1+L_2+L_4+2L_{12})$$
$$z_{22} = R_4+R_5+R_6+(S_4+S_5+S_6)/i\omega+i\omega(L_4+L_5+L_6-2L_{46})$$

$$z_{33} = R_1+R_3+R_5+(S_1+S_3+S_5)/i\omega + i\omega(L_1+L_3+L_5-2L_{13})$$
$$z_{44} = R_9+R_{10}+R_{11}+(S_9+S_{10})/i\omega$$
$$z_{55} = R_7+R_8+R_9+(S_8+S_9)/i\omega + i\omega(L_7+L_8-2L_{78})$$
$$z_{12} = R_4+S_4/i\omega + i\omega(L_4-L_{46}) = z_{21}$$
$$z_{13} = R_1+S_1/i\omega + i\omega(L_1-L_{13}+L_{12}-L_{23}) = z_{31}$$
$$z_{14} = z_{41} = z_{15} = z_{51} = z_{34} = z_{43} = z_{35} = z_{53} = z_{24} = z_{42} = 0$$
$$z_{23} = -R_5-S_5/i\omega - i\omega L_5 = z_{32}$$
$$z_{25} = i\omega L_{67} = z_{52}$$
$$z_{45} = -R_9 - S_9/i\omega .$$

Besides these equations we have also:

$$-J_1-J_2=I_4, \quad J_2-J_3=I_5, \quad J_2=I_6, \quad J_5=-I_8 \quad \text{(all known)}.$$

These results can be checked with the eqs. (20) of the example 5, of Ch. V, §1, by substituting the expressions of the first eleven equations into the last five and then eliminating $I_1, I_4, I_5, I_8, I_9, I_{11}$ (e.g.).

Actually, the complex canonical mesh equations of the network considered in this example are very easy to solve, because the network has a lot of current sources (the complex current through which are known). Thus, we may substitute:

$$J_1 = -I_4 - I_6, \quad J_2 = I_6, \quad J_3 = I_6 - I_5, \quad J_5 = -I_8 \quad \text{(all known)}$$

into the fourth of the complex canonical mesh equations given above and from the resulting equation we may find $J_4 = I_{10} = I_{11}$. Then from the first, third, and fifth, of the canonical equations we may find D_4, D_5, and D_8, respectively; and then from the second we obtain D_6. Finally, we can obtain all the complex currents I_k $(k=1,2,3,7,9,10,11)$ in terms of the J_m $(m=1,2,3,4,5)$, and then all the V_k in terms of the I_k and D_k $(k=4,5,6,8)$ by means of the first eleven eqs.(20), Ch. V, §1.

Example 3. Consider the network of the example given in Ch. III, §1 (p. 65), but now as a sinusoidal current network at the angular frequency $\omega > 0$, and let us use the same complete & independent set of $(n_m=9)$ meshes used there. The complex canonical mesh equations of the network are then (using the notation of example 3, Ch. V, §1):

$$\sum_{n=1}^{9} z_{mn} J_n = F_m \quad (m=1,2,\dots,9), \qquad J_5 = I_5 \quad \& \quad J_6 = I_6 \quad \text{(both known)}$$

where: $F_1 = E_1$, $F_2 = E_2$, $F_5 = D_5$, $F_6 = D_6$, $F_3 = F_4 = F_7 = F_8 = F_9 = 0$,

and $$z_{mn} = z_{nm} \quad (m \,\&\, n = 1,2,\dots,9),$$

and:

$$z_{11} = Z_{1,1} = Z_1 = R_1 + i\omega L_1 + S_1/i\omega$$

$$z_{22} = Z_{22} = Z_2 = R_2 + i\omega L_2 + S_2/i\omega$$

$$z_{33} = Z_3 + Z_7 - 2Z_{37} = R_3+R_7 + (S_3+S_7)/i\omega + i\omega(L_3+L_7 - 2|L_{37}|)$$

$$z_{44} = Z_4 + Z_8 - 2Z_{48} = R_4+R_8 + (S_4+S_8)/i\omega + i\omega(L_4+L_8 + 2|L_{48}|)$$

$$z_{55} = Z_5 + Z_9 - 2Z_{59} = R_5+R_9 + (S_5+S_9)/i\omega + i\omega(L_5+L_9 + 2|L_{59}|)$$

$$z_{66} = Z_6+Z_{10}-2Z_{6,10} = R_6+R_{10}+ (S_6+S_{10})/i\omega+i\omega(L_6+L_{10}-2|L_{6,10}|)$$

$$z_{77} = Z_7+Z_{10}+Z_{11} = R_7+R_{10}+R_{11}+(S_7+S_{10}+S_{11})/i\omega+i\omega(L_7+L_{10})$$

$$z_{88} = Z_8+Z_9+Z_{11} = R_8+R_9+R_{11} + (S_8+S_9+S_{11})/i\omega + i\omega(L_8 + L_9)$$

$$z_{99} = Z_8+Z_{10}+Z_{12} = R_8+R_{10}+R_{12}+(S_8+S_{10}+S_{12})/i\omega+i\omega(L_8+L_{10})$$

$$z_{12} = z_{14} = z_{15} = z_{16} = z_{17} = z_{18} = z_{19} = 0, \quad z_{13} = Z_{37} = -i\omega|L_{37}$$

$$z_{23} = z_{25} = z_{26} = z_{27} = 0, \qquad z_{24} = Z_{24} - Z_{28} = i\omega(|L_{24}|-|L_{28}|$$

$$z_{28} = Z_{28} = i\omega|L_{28}|, \qquad z_{29} = Z_{29} = i\omega|L_{28}|$$

$$z_{34} = z_{35} = z_{36} = z_{38} = z_{39} = 0, \quad z_{37} = -Z_7+Z_{37} = -R_7-S_7/i\omega-i\omega(L_7-$$

$$z_{45} = z_{46} = z_{47} = 0, \quad z_{48}=z_{49}=-Z_8+Z_{48}= -R_8-S_8/i\omega -i\omega(L_8+|L_{48}|$$

$$z_{56} = z_{57} = z_{59} = 0, \quad z_{58}=-Z_9+Z_{59} = -R_9-S_9/i\omega -i\omega(L_9+|L_{59}|$$

$$z_{67} = -Z_{10} + Z_{6,10} = -R_{10} - S_{10}/i\omega - i\omega L_{10} + i\omega|L_{6,10}|$$

$$z_{68} = 0, \quad z_{69} = Z_{10} - Z_{6,10} = R_{10} + S_{10}/i\omega + i\omega L_{10} - i\omega|L_{6,10}|$$

$$z_{78} = Z_{11} = R_{11}+S_{11}/i\omega, \qquad z_{79} = -Z_{10} = -R_{10}-S_{10}/i\omega - i\omega L_{10}$$

$$z_{89} = Z_8 = R_8 + S_8/i\omega + i\omega L_8.$$

These equations can be checked with those of example 3, Ch. V, §1, by substituting the V_k as given by the first twelve eqs. (13) of Ch. I §1, into the last nine, and then eliminating I_8, I_9 & I_{10} (say).

Problem 2. ***Establish the complex canonical mesh equations for the networks shown in figures 8 & 10 of Ch. V, §1.***

§4. THE SOLUTION OF THE CANONICAL MESH EQUATIONS.

In any particular case, the best way to solve the complex canonical mesh equations of a specific network, assuming that they are consistent, is to solve the eqs. (6), §1, for n_{cs} of the complex mesh currents in terms of the (known) complex currents through the current sources and the other complex mesh currents, and then to eliminate them by substitution in the eqs. (2), §2. In this way we obtain a system of n_m equations in n_m unknowns. By solving these n_m equations for the n_m unknowns left in them and then reversing the previous steps all the unknown complex mesh currents J_m $(m = 1, 2, \ldots, n_m)$ and complex voltage rises D_k $(k = 1, 2, \ldots, n_{cs})$ across the current sources can be obtained. They should then be substituted into the $n_m + n_{cs}$ complex canonical mesh equations (3), §2, to check the results.

Once all the complex mesh currents J_m and all the complex voltage rises D_k have been obtained (and checked), all the complex currents I_k $(k = 1, 2, \ldots, n_e)$ through the general series elements of the network may be obtained by means of eqs. (1) of the introduction to this chapter; and then all the complex voltage drops V_k $(k = 1, 2, \ldots, n_e)$ across the general series elements may be obtained by eqs. (2), §1.

Finally, all the complex currents and voltages may be transformed back to sine functions of the same angular frequency as that of the sources, as explained in Ch. V, §1(p.130), by means of the fundamental isomorphism between complex numbers and sinusoids of the same frequency; but this step is usually omitted because it is considered trivial.

Problem 1. Consider the system of n equations:

$$\sum_{l=1}^{n} a_{kl} x_l = b_k \qquad (k=1,2,\ldots,n) \tag{1}$$

in the n unknowns x_l. Assume that the determinant, $\det(a_{kl}) = \det$, of the coefficients is not zero. Denoting the cofactor of a_{kl} divided by the determinant by $c_{lk} = \operatorname{cof} a_{kl}/\det$, show that we have:

$$x_l = \sum_{k=1}^{n} c_{lk} b_k = \sum_{k=1}^{n} b_k \operatorname{cof} a_{kl}/\det. \qquad (l=1,2,\ldots,n) \tag{2}$$

[Hint: Make use of the fact that:

$$\sum_{k=1}^{n} c_{hk} a_{kl} = \sum_{k=1}^{n} a_{kl} \operatorname{cof} a_{kh}/\det = \sum_{k=1}^{n} a_{hk} c_{kl} = \sum_{k=1}^{n} a_{hk} \operatorname{cof} a_{lk}/\det = \delta_{hl}, \tag{3}$$

where $\delta_{hl} = 1$ if $h = l$, and $\delta_{hl} = 0$ if $h \neq l$ (Kronecker's delta).]

When the number n_{cs} of current sources in a given network is zero, and the determinant, $\det = \det(z_{mn})$, of the mesh impedances is not zero, the corresponding complex canonical mesh equations (4), §2, namely:

$$\sum_{n=1}^{n_m} z_{mn} J_n = \sum_{k=1}^{n_e} [k,m] E_k = \tilde{E}_m \qquad (m = 1, 2, \ldots, n_m) \tag{4}$$

can be solved for the complex mesh currents in the following form:

$$J_n = \sum_{m=1}^{n_m} y_{nm} \tilde{E}_m = \sum_{m=1}^{n_m} \tilde{E}_m \operatorname{cof} z_{mn}/\det = \sum_{m=1}^{n_m} \sum_{k=1}^{n_e} E_k [k,m] \operatorname{cof} z_{mn}/\det, \tag{5}$$

where: $y_{mn} = \operatorname{cof} z_{nm}/\det$, for m & $n = 1, 2, \ldots, n_m$. (6)

And when $n_{cs} > 0$, the corresponding complex canonical mesh equations (3), §2, namely:

$$\sum_{n=1}^{n_m} z_{mn} J_n - \sum_{k=1}^{n_{cs}} [k,m] D_k = \sum_{k=1}^{n_e} [k,m] E_k = \tilde{E}_m \qquad (m = 1, 2, \ldots, n_m) \tag{7}$$

$$\sum_{n=1}^{n_m} [l,n] J_n = I_l \text{ (known)} \, (l = 1, 2, \ldots, n_{cs} \leq n_m) \tag{8}$$

(assuming the elements with current sources to be numbered first, with the numbers $1,2,\ldots,n_{cs}$, and the others last: $n_{cs}+1,\ldots,n_e$), can be solved for the complex mesh currents and voltage rises in the following way.

Let us define the following quantities:

$$\tilde{I}_l = \sum_{n=1}^{n_m}\sum_{m=1}^{n_m}[l,n]\,y_{nm}\tilde{E}_m = \sum_{n=1}^{n_m}\sum_{m=1}^{n_m}\sum_{k=1}^{n_e}[l,n]\,y_{nm}[k,m]E_k \qquad (l=1,2,\ldots,n_{cs})$$

$$\tilde{y}_{lk} = \sum_{n=1}^{n_m}\sum_{m=1}^{n_m}[l,n]\,y_{nm}[k,m] \qquad (l \,\&\, k = 1,2,\ldots,n_{cs})$$

$$\tilde{z}_{kl} = \operatorname{cof}\tilde{y}_{lk}/\det' \qquad (\text{assuming } \det' = \det(\tilde{y}_{kl}) \neq 0)$$

Then we will have from eqs. (7) that:

$$J_n = \sum_{m=1}^{n_m} y_{nm}\Big(\tilde{E}_m + \sum_{k=1}^{n_{cs}}[k,m]D_k\Big) \qquad (n = 1,2,\ldots,n_m)$$

where the y_{mn} are given by eqs. (6), assuming $\det = \det(z_{mn}) \neq 0$, as bef

Hence, substituting these expressions in eqs. (8), we get:

$$\sum_{n=1}^{n_m}[l,n]J_n = \sum_{n=1}^{n_m}\sum_{m=1}^{n_m}[l,n]\,y_{nm}\Big(\tilde{E}_m + \sum_{k=1}^{n_{cs}}[k,m]D_k\Big) = I_l \ ; \ (l = 1,2,\ldots,n_{cs}$$

that is:

$$\tilde{I}_l + \sum_{k=1}^{n_{cs}}\tilde{y}_{lk}D_k = I_l \ , \qquad (l=1,2,\cdots,n_{cs}) \qquad (1$$

so that:

$$D_k = \sum_{l=1}^{n_{cs}}\tilde{z}_{kl}(I_l - \tilde{I}_l). \qquad (k = 1,2,\ldots,n_{cs}) \qquad (1$$

Hence, substituting in eqs. (12), we obtain:

$$J_n = \sum_{m=1}^{n_m} y_{nm}\Big\{\tilde{E}_m + \sum_{k=1}^{n_{cs}}\sum_{l=1}^{n_{cs}}[k,m]\,\tilde{z}_{kl}(I_l - \tilde{I}_l)\Big\}. \qquad (n=1,2,\ldots,n_m)$$

Problem 2. Check that the J_n given by eqs. (5) satisfy eqs. (4, and that the J_n and D_k given by eqs. (15 & 14) satisfy eqs. (7 & 8).

Problem 3. If $p = \sigma + i\omega$ is a generalized natural frequency (also called a natural complex fr of a network, the general equations of the network with all the exciting functions nullified shall be satisfied by response functions of the form $A\,e^{\sigma t}\sin(\omega t+\alpha)$, or of the form $A\exp(pt)$. Show that the generalized natural frequencies of an arbitrary (stationary) constant parameter network of general series elements are given by the solutions, p, of the following determinantal equation (assuming that not every p is a solution, since otherwise this would be meaningless), where $z_{mn} = z_{mn}(p)$ are the mesh impedances at the generalized frequency p, namely:

$$z_{mn}(p) = r_{mn} + p\,l_{mn} + \frac{s_{mn}}{p}.$$

(see eqs. 7, of §2).

(Cf. Prob. 21, Ch. V, §

$$\begin{array}{c} \\ \uparrow \\ n_m \\ \downarrow \\ \uparrow \\ n_{cs} \\ \downarrow \\ \\ \end{array}
\begin{array}{|cccccccc|}
z_{11} & z_{12} & \cdots & z_{1n_m} & [1,1] & [2,1] & \cdots & [n_{cs},1] \\
z_{21} & z_{22} & \cdots & z_{2n_m} & [1,2] & [2,2] & \cdots & [n_{cs},2] \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
z_{n_m1} & \cdots & & z_{n_mn_m} & [1,n_m] & \cdots & \cdots & [n_{cs},n_m] \\
[1,1] & [1,2] & \cdots & [1,n_m] & 0 & 0 & \cdots & 0 \\
[2,1] & [2,2] & \cdots & [2,n_m] & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
[n_{cs},1] & \cdots & & [n_{cs},n_m] & 0 & \cdots & \cdots & 0 \\
\hline
\multicolumn{4}{c}{\leftarrow n_m \rightarrow} & \multicolumn{4}{c}{\leftarrow n_{cs} \rightarrow}
\end{array}
=
\begin{array}{c} \uparrow \\ n_m \text{ rows} \\ \downarrow \\ \uparrow \\ n_{cs} \text{ rows} \\ \downarrow \\ \\ \end{array}
\begin{array}{|c|c|}
z_{mn}(p) & [k,m]_{\text{Transposed}} \\
(m \,\&\, n=1,2,\ldots,n_m) & (k=1,2,\ldots,n_{cs}) \\
 & (m=1,2,\ldots,n_m) \\
[k,n] & \\
(k=1,2,\ldots,n_{cs}) & 0 \\
(n=1,2,\ldots,n_m) & \\
\hline
\leftarrow n_m \text{ columns} \rightarrow & \leftarrow n_{cs} \text{ columns} \rightarrow
\end{array}
= 0.$$

(When $n_{cs}=0$, the generalized natural frequencies are given by the solutions p of: $\det(z_{mn}(p)) = |(z_{mn}(p))| = 0$, of course.)

§5. EXISTENCE AND UNIQUENESS THEOREMS.

The restricted alternating current problem of a network (see Ch. V, §1, p. 115), i.e. the problem of finding the response sinusoidal currents and voltages of the same frequency as that of all the exciting sinusoidal functions of the sources of the network, shall have a solution if and only if the complex equations of the network have one. This is so because the isomorphism (i.e. the one-to-one correspondence) between complex numbers and sinusoids of the same frequency preserves linear combinations and equations, and carries the equations of the network in the sinusoidal state into the complex equations of the network, and viceversa. Moreover, due to the reversible character of the transformation of the complex equations of a network into its complex canonical equations, it is clear that they are equivalent systems, and either system shall have a solution if and only if the other has one. Furthermore, it is obvious that if all the unknown complex circulating currents J_m $(m=1,2,\ldots,n_m)$ *and all the unknown complex voltage rises* D_k $(k=1,2,\ldots,n_{cs})$ *through the current sources are unique, then so are all the complex currents* I_k *and all the complex voltage drops* V_k $(k=1,2,\ldots,n_e)$ *through the elements of the network. And the converse of this is also true: if all the* V_k *and* I_k *are unique then so are all the* J_m *and* D_k*. Because if there were two sets of* J_m *and* D_k *corresponding to a unique set of* I_k *and* V_k*, then by simple subtraction we could find from eqs. (2), §1, that the corresponding* D_k *are equal; and from the eqs. (1) of the introduction to this chapter we could obtain:*

$$\sum_{m=1}^{n_m}[k,m]\Delta J_m = 0, \qquad (k = 1,2,\ldots,n_e) \qquad (1)$$

where the ΔJ_m *are the differences between corresponding* J_m *of the two sets. But the coefficients in this system of equations are the same as those in Kirchhoff's voltage laws (eqs. 4, §1) for a complete and independent set of meshes of the network, which are known to be linearly independent (cf. note of Ch. III, §1, p. 64); consequently from:*

$$\sum_{k=1}^{n_e}\sum_{m=1}^{n_m}[k,m]\,V_k\,\Delta J_m = \sum_{m=1}^{n_m}\Big(\sum_{k=1}^{n_e}[k,m]\,V_k\Big)\,\Delta J_m = 0,$$

(which is obtained from eqs. (1) by multiplying the $k^{\underline{th}}$ *equation by* V_k*, considered as an indeterminate, and adding the* n_e *results) we infer that all the* ΔJ_m $(m = 1,2,\ldots,n_m)$ *are zero (see Ch. II, §4, 2º, p. 56).*

Now it is a well known theorem of algebra (see e.g. M. Bôcher, Higher Algebra, Macmillan Co., p. 43) that the complex canonical equa-

tions of a given network shall have a solution for arbitrary source values if and only if the determinant of the coefficients of the unknowns does not vanish (in which case, the solution is also unique for any given source values); that is, when $n_{cs} > 0$ (by eqs. 7 & 8, §4), if and only if:

$$\Delta = \begin{vmatrix} z_{11}(i\omega) & z_{12}(i\omega) & \cdots & z_{1n_m}(i\omega) & [1,1] & [2,1] & \cdots & [n_{cs},1] \\ z_{21}(i\omega) & z_{22}(i\omega) & \cdots & z_{2n_m}(i\omega) & [1,2] & [2,2] & \cdots & [n_{cs},2] \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ z_{n_m1}(i\omega) & z_{n_m2}(i\omega) & \cdots & z_{n_mn_m}(i\omega) & [1,n_m] & [2,n_m] & \cdots & [n_{cs},n_m] \\ [1,1] & [1,2] & \cdots & [1,n_m] & 0 & 0 & \cdots & 0 \\ [2,1] & [2,2] & \cdots & [2,n_m] & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ [n_{cs},1] & [n_{cs},2] & \cdots & [n_{cs},n_m] & 0 & 0 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} (z_{mn}) & ([k,m])_{\text{Transposed}} \\ ([k,m]) & (0) \end{vmatrix} \neq 0 ; \qquad (2)$$

and when $n_{cs} = 0$, if and only if: $\det (z_{mn}(i\omega)) = |(z_{mn})| \neq 0$ (by eqs. 4, §4); where the $z_{mn}(i\omega)$ are given by eqs. (6 & 7), or by eqs. (1), of §2. Note that ω can then not be a natural angular frequency (i.e. $0 + i\omega = i\omega$ can then not be a generalized natural frequency) of the network. Networks for which ~~eq.~~ relation (2) holds are called non-singular networks; and when no mention is made to the contrary, we shall always assume the networks considered to be non-singular. REMARK: This, however, is only a theoretical limitation; because in any actual network, all elements have some resistance, inductance, or capacitance, and the practical sources are not ideal, and it is clear that the currents and voltages through its elements will exist and be unique (even if it does burn), for any given sources, so that relation (2) will hold when the network is properly represented by taking into consideration all these things. (See the note at the end of §7 of Ch. XII.)

When the relation (2) holds for a given network, the unique values of the unknown complex mesh currents J_m and current source complex voltage rises D_k can be obtained by the well-known and elementary rule of Cramer, namely:

$$J_m = \Delta_m / \Delta \quad (m = 1, 2, \dots, n_m), \qquad -D_k = \Delta_{n_m+k} / \Delta \quad (k = 1, 2, \dots, n_{cs}), \qquad (3)$$

where Δ_l $(l = 1, 2, \dots, n_m, n_m+1, \dots, n_m+n_{cs})$ is the determinant obtained from Δ by replacing the elements of the $l^{\underline{th}}$ column by the elements $\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{n_m}, I_1, I_2, \dots, I_{n_{cs}}$ (i.e. by replacing the $l^{\underline{th}}$ column in Δ by the independent terms in the eqs. 7 & 8 of §4). In particular, (*): when all the sources have null values, the only possible solution is the trivial null solution (with all the unknowns = 0); because all the Δ_l are then zero, since each will then have a column of zeros.

When the relation (2) does not hold for a given network, let r be the order (=number of rows) of the non-vanishing determinant of highest order that can be extracted from the following table, or rectangular array:

$$\begin{array}{cccccccc}
z_{11} & z_{12} & \cdots & z_{1n_m} & [1,1] & [2,1] & \cdots [n_{cs},1] & \tilde{E}_1 \\
z_{12} & z_{22} & \cdots & z_{2n_m} & [1,2] & [2,2] & \cdots [n_{cs},2] & \tilde{E}_2 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
z_{n_m1} & z_{n_m2} & \cdots & z_{n_mn_m} & [1,n_m] & [2,n_m] & \cdots [n_{cs},n_m] & \tilde{E}_{n_m} \\
[1,1] & [1,2] & \cdots & [1,n_m] & 0 & 0 & \cdots\ 0 & I_1 \\
[2,1] & [2,2] & \cdots & [2,n_m] & 0 & 0 & \cdots\ 0 & I_2 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
[n_{cs},1] & [n_{cs},2] & \cdots & [n_{cs},n_m] & 0 & 0 & \cdots\ 0 & I_{n_{cs}}
\end{array} \tag{4}$$

by the deletion of columns and rows (called the rank of the table), and let r' be the rank of the table when the last column is omitted. Then $r' \leqq r \leqq n_m+n_{cs}$; and it is also a well-known result of algebra (see Bôcher, p. 44) that the complex canonical equations (4, or 7 & 8, of §4) of the network shall ~~then~~ have a solution if and only if $r=r'$; and if $r=r'<n_m+n_{cs}$ then r of the equations (with a non-vanishing determinant of order r formed with the coefficients of r of the unknowns) are linearly independent while the rest are linearly dependent on them, and so the values of (the other) $n_m+n_{cs}-r$ unknowns can by assigned at will and then the other r unknowns can be determined uniquely in terms of them from the r linearly independent equations, by means of Cramer's rule. The unknowns thus found will satisfy all the n_m+n_{cs} equations.

In other words, when the relation (2) does not hold for a given network, the complex canonical equations of the network shall have a solution only for particular complex source values, and that (if and) only if they are linearly dependent for the given complex source values (i.e. if and only if the rows in the table (4), considered as coefficients of n_m+n_{cs} linear forms in $n_m+n_{cs}+1$ indeterminates or arbitrary variables, are linearly dependent); and when they are linearly dependent, the maximum number of linearly independent ones (called the rank of the system) is the same as the rank, r, of the table (4), and the values of some set of $n_m+n_{cs}-r$ unknowns can be assigned at will and then the other r unknowns can be determined in terms of them. In any case when (2) does not hold, the complex equations of the network shall have infinitely many solutions if they have any, and so a solution, if it exists, can not be unique at any rate.

In the above, we have explicit criteria for the existence and uniqueness of the solution of the restricted alternating current problem (i.e. the sinusoidal problem) of any given network, in terms of the parameters (via the mesh impedances) and the combinatorial structure (via the incidence numbers) of the network. However, even *though the criteria and solution formulae based on the evaluation of determinants are very neat, compact, and of great theoretical importance, for practical purposes they are clumsy and can not be recommended (especially if the number of equations is large, say four or more). In the appendix to this chapter we will develop, along the lines of linear dependency, one of the finest methods known to date of treating a system of linear equations, based on an article by* <u>*P.D. Crout*</u>*: A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients, Transactions of the American Institute of Electrical Engineers, Vol. 60 (1941), pp. 1235-1241.* Also see the article by <u>Paul S. Dwyer</u>: The solution of simultaneous equations, Psychometrika, (1941) pp. 101-129.

According to § 4, when det $(z_{mn}) \neq 0$ *and, if* $0 < n_{cs} \leqq n_m$ *(cf. note in Ch. III, §1, p. 64), when det* $(\tilde{y}_{kl}) \neq 0$*, where the mesh impedances* z_{mn} $(m \,\&\, n = 1,2,\ldots,n_m)$ *are given by eqs. (1), §2, and the* $\tilde{y}_{kl}$ $(k \,\&\, l = 1,2,\ldots,n_{cs} \leqq n_m)$ *are given by eqs. (10), §4, in terms of the* $y_{mn} = \operatorname{cof} z_{nm} / \det(z_{mn})$ $(m \,\&\, n = 1,2,\ldots,n_m)$*, the solution of the complex canonical mesh equations of a given network is given by eqs. (5), §4, when* $n_{cs}=0$*, or by eqs. (14 & 15), when* $n_{cs}>0$*, of § 4, and so exists and is unique for arbitrary complex source values, and hence relation (2) then holds. However, the converse of this is not necessarily true, unless* $n_{cs}<n_m$*. Because if* $n_{cs}=n_m$ *then* $-\Delta = \{\det([k,m])\}^2 \neq 0$*, as can be seen from eq. (2), by developing the determinant* Δ *according to the last* n_m *columns by the method of Laplace (see Bocher, Higher Algebra, p. 24), whether det* (z_{mn}) *vanishes or not. On the other hand, if* $n_{cs}<n_m$ *and* $\Delta \neq 0$*, then all the complex canonical equations are consistent and linearly independent, in the first place, and, in the second place, if det* (z_{mn}) *were zero, we could multiply the* $m^{\underline{th}}$ *equation (7), § 4, by cof* z_{mp} *and then add over all* $m = 1,2,\ldots,n_m$ *(see the hint to Prob. 1, §4) to obtain:*

$$-\sum_{m=1}^{n_m}\sum_{k=1}^{n_{cs}} D_k[k,m] \operatorname{cof} z_{mp} = \sum_{m=1}^{n_m} E_m \operatorname{cof} z_{mp}\,, \quad (p = 1,2,\ldots,n_m)$$

and these n_m *equations, which have only* n_{cs} *unknowns, would either be inconsistent or linearly dependent, and so the first* n_m *eqs. (7), § 4, would either be inconsistent or linearly dependent, which is a contradiction; hence we must have: det* $(z_{mn}) \neq 0$*. Once this is established, one can then carry out the steps from eqs. (7 & 8) to eqs. (13), of § 4, and since the latter must also be linearly independent if* $\Delta \neq 0$*, we must also have det* $(\tilde{y}_{kl}) \neq 0$ *when* $n_{cs} < n_m$*. In short, when* $n_{cs}<n_m$*,* $\Delta \neq 0$ *is equivalent to det* $(z_{mn}) \neq 0$ *& det* $(\tilde{y}_{kl}) \neq 0$*.*

§6. THE SUPERPOSITION PRINCIPLE.

Consider any non-singular (stationary) constant parameter network with n_e general series elements, n_m independent meshes, and n_{cs} ($\leq n_m$) current sources (in elements assumed to be numbered first), assumed to be in the sinusoidal state at the angular frequency $\omega \neq 0$. We have seen in §5 that the complex canonical mesh equations of the network have an unique solution given by Cramer's rule, expressed in eqs. (3), §5. If we now develop each of the determinants Δ_l ($l = 1,2,\ldots,n_m,\ldots,n_m+n_{cs}$) according to the corresponding $l^{\underline{th}}$ column [formed by the elements $\tilde{E}_1,\ldots,\tilde{E}_{n_m}$, $I_1,\ldots,I_{n_{cs}}$, where the $\tilde{E}_m$ ($m = 1,2,\ldots,n_m$) are given in terms of the complex electromotive forces E_k ($k = 1,2,\ldots,n_e$) of the voltage sources in the network by eqs. (7), §4, and the I_k ($k = 1,2,\ldots,n_{cs}$) are the complex currents through the current sources in the network] we will obtain the following equations:

$$
\begin{aligned}
J_m &= \sum_{n=1}^{n_m} a'_{mn}\tilde{E}_n + \sum_{l=1}^{n_{cs}} a_{ml}I_l = \sum_{n=1}^{n_m}\sum_{l=1}^{n_e} a'_{mn}[l,n]E_l + \sum_{l=1}^{n_{cs}} a_{ml}I_l ,\\
D_k &= \sum_{n=1}^{n_m} b'_{kn}\tilde{E}_n + \sum_{l=1}^{n_{cs}} b_{kl}I_l = \sum_{n=1}^{n_m}\sum_{l=1}^{n_e} b'_{kn}[l,n]E_l + \sum_{l=1}^{n_{cs}} B_{kl}I_l ,
\end{aligned} \qquad (1)
$$

where the a'_{mn}, a_{ml}, b'_{kn}, b_{kl} (m & $n = 1,2,\ldots,n_m$ and k & $l = 1,2,\ldots,n_{cs}$) are certain cofactors in the determinant Δ (of relation (2), §5), divided by Δ, whose explicit expressions are irrelevant at present.

Now if we interchange the orders in the double summations in eqs. (1), and if we put (for $l = 1,2,\ldots,n_e$):

$$
A_{ml} = \sum_{n=1}^{n_m} a'_{mn}[l,n] \qquad \text{and} \qquad B_{kl} = \sum_{n=1}^{n_m} b'_{kn}[l,n]
$$

we will obtain:

$$
\begin{aligned}
J_m &= \sum_{l=1}^{n_e} A_{ml}E_l + \sum_{l=1}^{n_{cs}} a_{ml}I_l , \quad (m = 1,2,\ldots,n_m)\\
D_k &= \sum_{l=1}^{n_e} B_{kl}E_l + \sum_{l=1}^{n_{cs}} b_{kl}I_l , \quad (k = 1,2,\ldots,n_{cs})
\end{aligned} \qquad (2)
$$

where the coefficients A_{ml}, a_{ml}, B_{kl}, b_{kl} depend only on the parameters of the network, its combinatorial structure, and on the angular frequency ω, but otherwise are independent of the sources.

Before proceeding, we wish to say a few words about the concept of nullifying a source. To nullify a source means to replace it by a similar source but of value zero. Thus to nullify a voltage source means to replace it by a voltage source of zero electromotive force. This is like replacing it by a short-circuit (which can be done by throwing the switch in Fig. 1 from position 1 to 2)

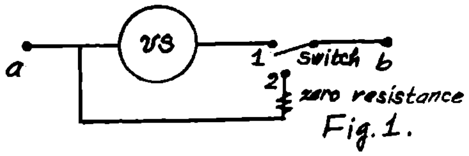


Fig.1.

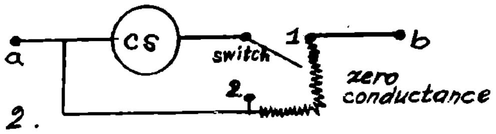


Fig.2.

but without identifying its terminals; or better, by replacing the voltage source by an element of zero resistance (or of zero impedance). To nullify a current source means to replace it by a current source obliged not to allow any current to flow through it. This is like simply removing the current source or replacing it by an open circuit, but without losing track of the circuit element across which one may speak of a voltage rise; or bettter, by replacing the current source by an element of zero conductance (or zero admittance), which can be done by throwing the switch in Fig. 2 from position 1 to 2.

Now when all the sources in a given network are nullified except a typical voltage source in element $\underline{l}$, *then (by eqs. 2) the complex circulating currents in the corresponding reduced (auxiliary) network are:* $J_m = A_{ml}E_l$ $(m = 1,2,\ldots,n_m)$, *and the complex voltage rises appearing across the places where the current sources were connected are:* $D_k = B_{kl}E_l$ $(k = 1,2,\ldots,n_{cs})$. *Likewise, when all the sources except a typical current source in element* $\underline{l}$ $(1 \leq l \leq n_{cs})$ *are nullified, then the complex circulating currents in the corresponding reduced (auxiliary) network are:* $J_m = a_{ml}I_l$ $(m = 1,2,\ldots,n_m)$, *and the complex voltage rises appearing across the places where the current sources were connected are:* $D_k = b_{kl}I_l$ $(k = 1,2,\ldots,n_{cs})$. *Moreover, the coefficients in the eqs. (2) do not depend on the sources (except for their common frequency), and so they are the same in all the auxiliary networks as in the original (given) network, taking into consideration that, due to the proper concept of source nullification, the combinatorial structures are also the same. Accordingly, by eqs. (2), we see that the complex circulating currents* J_m $(m = 1,2,\ldots,n_m)$ *and the complex voltage rises* D_k $(k = 1,2,\ldots,n_{cs})$ *across the current sources in the original network can be obtained by superposing (i.e. adding) the corresponding quantities of the auxiliary netowrks, obtained from the original network by nullifying all sources except one at a time. This result is called the Superposition Principle for non-singular networks.*

Problem 1. With the help of eqs. (1) of the introduction to this chapter and eqs. (2), § 1, show that the complex currents I_k *and the complex voltage drops* V_k $(k=1,2,\ldots,n_e)$ *through the elements of the given network can also be obtained by superposition.*

Another form of the superposition principle can be obtained as follows. Imagine all the sources of a given non-singular network arbitrarily grouped into two classes <u>*K*</u> *and* <u>*K'*</u>, *and let us split the summations in eqs. (2) accordingly, as follows:*

$$
\begin{aligned}
J_m &= \Big(\sum_{l\in K_{(vs)}} A_{ml}E_l + \sum_{l\in K_{(cs)}} a_{ml}I_l\Big) + \Big(\sum_{l\in K'_{(vs)}} A_{ml}E_l + \sum_{l\in K'_{(cs)}} a_{ml}I_l\Big), \quad (m=1,2,\dots,n_m) \\
D_k &= \Big(\sum_{l\in K_{(vs)}} B_{kl}E_l + \sum_{l\in K_{(cs)}} b_{kl}I_l\Big) + \Big(\sum_{l\in K'_{(vs)}} B_{kl}E_l + \sum_{l\in K'_{(cs)}} b_{kl}I_l\Big), \quad (k=1,2,\dots,n_{cs})
\end{aligned}
\qquad (3)
$$

where $\sum_{l\in K}$, *and* $\sum_{l\in K'}$, *mean summations over those terms with* <u>*l*</u> *belonging to a source of the classes K, and K', respectively. Now by what was said above, it is easy to see that the terms in the first parentheses in these equations represent the complex circulating currents, and the complex voltage rises across the current sources, in an auxiliary network obtained from the given network by nullifying all the sources of the class K' only, and the terms in the second parentheses in these equations represent the corresponding quantities in another auxiliary network obtained from the original network by nullifying all the sources of the other class* <u>*K*</u> *only. Hence, according to eqs. (3), we see that the complex currents and voltages in a given non-singular network can be obtained by superposing (i.e. adding) corresponding quantities in the two auxiliary networks obtained from the given network, one, by nullifying only all the sources of one of two classes into which all the sources of the given network are arbitrarily grouped, and the other, by nullifying all the sources of the other class only. (This result can naturally be extended to any number of mutually exclusive classes by further splitting these two classes.) This is a* <u>*second form of the superposition principle*</u>.

The superposition principle can also be generalized to (not non-)singular constant parameter networks; but besides superposing corresponding complex currents and voltages in the auxiliary networks obtained from a given network by nullifying some of the sources at a time, only, it is also necessary to add the corresponding quantities in another auxiliary network obtained from the given network by nullifying all its sources simultaneously (because sinusoidal currents and voltages of the same frequency as that of the sources may spontaneously build up in it) in order to obtain the complex currents and voltages in the given network. However, the details are long and will not be given here.

<u>*Note*</u>. *From eqs. (1), or (2), we see that the complex currents and voltages in the elements of a non-singular constant parameter network are homogeneous linear combinations (without independent terms) of the complex current and voltage source values (of essential elements only). For singular networks, independent terms must be added.*

THE DWYER-CROUT METHOD OF SUCCESSIVE ELIMINATION.

Consider a system of $\underline{m}$ linear equations in $\underline{n}$ unknowns $(m \leqq n)$:

$$\sum_{l=1}^{n} a_{kl} x_l = a_k = a_{k,n+1} . \qquad (k = 1,2,\ldots,m) \qquad (1)$$

The Dwyer-Crout method (see the references given in §5, page 181) consists in reducing this system to an equivalent triangular system:

$$x_k + \sum_{l=k+1}^{n} b_{kl} x_l = b_k = b_{k,n+1} , \qquad (k = 1,2,\ldots,m) \qquad (2)$$

where the coefficients and independent terms b_{kl} are computed successively by the following scheme:

$$b_{k1} = a_{k1} \ (k = 1,2,\ldots,m), \qquad b_{1l} = a_{1l}/a_{11} = a_{1l}/b_{11} \ (l = 2,\ldots,n+1)$$

$$b_{kl} = a_{kl} - \sum_{h=1}^{l-1} b_{kh} b_{hl} , \quad \text{for} \quad l \leqq k \leqq m \qquad (3)$$

$$b_{kl} = (a_{kl} - \sum_{h=1}^{k-1} b_{kh} b_{hl})/b_{kk} , \quad \text{for} \quad k < l \leqq n+1,$$

assuming that (all goes well and that) all the $b_{kk} \neq 0$ $(k = 1,2,\ldots,m)$.

All the solutions can then be found in backward succession from:

$$x_k = b_k - \sum_{l=k+1}^{n} b_{kl} x_l , \qquad (k = m, m-1, \ldots, 2, 1) \qquad (4)$$

(obtained from eqs. (2) by solving for x_k in terms of the succeeding x_k) where the last $(n-m)$ unknowns may be given arbitrary values (when $m<n$).

To prove this, let us introduce the following notation:

$$b^*_{kl} = \begin{cases} 0, & \text{if } k>l \\ 1, & \text{if } k = l \\ b_{kl}, & \text{if } k<l \end{cases} \qquad (k=1,2,\ldots,m; \quad l=1,2,\ldots,n+1). \qquad (5)$$

Then the eqs. (3) can be put into the following recursive scheme:

$$\begin{aligned} a_{kl} &= \sum_{h=1}^{l} b_{kh} b^*_{hl} = \sum_{h=1}^{k} b_{kh} b^*_{hl} , \quad \text{for } l \leqq k \leqq m \ (\text{since } b^*_{hl} = 0 \text{ for } h>l) \\ a_{kl} &= \sum_{h=1}^{k} b_{kh} b_{hl} = \sum_{h=1}^{k} b_{kh} b^*_{hl} , \text{ for } k<l \leqq n+1, \ (\text{since } b_{hl} = b^*_{hl} \end{aligned} \qquad (6)$$

for $h \leqq k < l$). Thus for all $k = 1,\ldots,m$ and $l = 1,\ldots, n, n+1$, we have:

$$a_{kl} = \sum_{h=1}^{k} b_{kh} b^*_{hl} . \qquad (6)$$

With the help of these equations, the first members of the given system (1) can be transformed as follows:

$$\begin{aligned} \sum_{l=1}^{n} a_{kl} x_l &= \sum_{l=1}^{n} \sum_{h=1}^{k} b_{kh} b^*_{hl} x_l = \sum_{h=1}^{k} b_{kh} \sum_{l=1}^{n} b^*_{hl} x_l \\ &= \sum_{h=1}^{k} b_{kh} (x_h + \sum_{l=h+1}^{n} b_{hl} x_l), \qquad (k = 1,2,\ldots,m), \end{aligned} \qquad (7)$$

since b^*_{hl} is 0 for $l<h$ and it is 1 for $l = h$, and $b^*_{hl} = b_{hl}$ for $l>h$, by eqs. (5);

and for the second members of eqs. (1) we have (by the 2nd eqs. 6):

$$a_k = a_{k,n+1} = \sum_{h=1}^{k} b_{kh} b_{h,n+1} = \sum_{h=1}^{k} b_{kh} b_h \,. \quad (k=1,2,\ldots,m) \qquad (8)$$

Hence, denoting the differences of the first and second members of eqs. (1) by A_k $(k=1,2,\ldots,m)$, and the differences of the first and second members of eqs. (2) by B_k, we have (by eqs. 7 & 8):

$$A_k = \sum_{l=1}^{n} a_{kl} x_l - a_k = \sum_{h=1}^{k} b_{kh} \sum_{l=1}^{n} b^*_{hl} x_l - \sum_{h=1}^{k} b_{kh} b_h \quad (k=1,2,\ldots,m)$$

$$= \sum_{h=1}^{k} b_{kh} \Big(\sum_{l=1}^{n} b^*_{hl} x_l - b_h\Big) = \sum_{h=1}^{k} b_{kh}\Big(x_h + \sum_{l=h+1}^{n} b_{hl} x_l - b_h\Big) = \sum_{h=1}^{k} b_{kh} B_h \,. \qquad (9)$$

From these equations we see that the A_k are linear combinations of the B_k; in fact, A_k is a linear combination of $B_1,\ldots,B_k$ only. Moreover, when all the $b_{kk} \neq 0$ $(k=1,2,\ldots,m)$, from eqs. (9) we can easily obtain the B_k as linear combinations of the A_k; in fact, B_k results a linear combination of $A_1,\ldots,A_k$ only. Consequently, when all the B_k vanish so shall the A_k vanish also, and conversely when all the $b_{kk} \neq 0$. In other words, every solution of eqs. (1) shall also be a solution of eqs. (2), and conversely under the assumed conditions, so that eqs. (1) and eqs. (2) are equivalent systems.

Finally, eqs. (4) are obtained from eqs. (2) by solving them for x_k in terms of the succeeding x_l $(l=m+1,\ldots,n)$. When $m<n$, the last $n-m$ unknowns: $x_{m+1},\ldots,x_n$, can be assigned arbitrarily and then x_m can be found in terms of the assigned values by eq. (4) for $k=m$. Then x_{m-1} can be found in terms of $x_m,\ldots,x_n$ by eq. (4) for $k=m-1$; and so on, till finally x_1 is found in terms of $x_2,\ldots,x_n$ by eq. (4) for $k=1$. This completes the proof of the method.

<u>Problem 1</u>. From eqs. (9), show that the B_k $(k=1,2,\ldots,m)$ are given in terms of the A_k by the following recursive scheme:

$$B_1 = A_1/b_{11} \quad \& \quad B_k = \Big(A_k - \sum_{l=1}^{k-1} b_{kl} B_l\Big)/b_{kk} \quad (k=1,2,\ldots,m). \qquad (10)$$

In practice, the method is carried out by tabulating the coefficients and independent terms of the given equations (1), in a rectangular array, as follows:

$$\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & a_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & a_m \end{array} \qquad (11)$$

and then constructing the table of the quantities b_{kl} defined by the recursive scheme (3), as follows:

$$\begin{array}{ccccc} b_{11} & b_{12} & \ldots & b_{1n} & b_1 \\ b_{21} & b_{22} & \ldots & b_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{m1} & b_{m2} & \ldots & b_{mn} & b_m \end{array} \qquad (12)$$

This table coincides with the table of coefficients and independent terms of the eqs. (2) only in the entries above and to the right of the principal diagonal (formed by the entries b_{11}, b_{22}, ...) while the corresponding table for the eqs. (2), which is precisely that of the quantities b^*_{kl}, has ones on the principal diagonal and zeros below; however, the entries on and below the principal diagonal in table (12) are made for computational purposes and may be ignored at the end when eqs. (2), or (what is the same thing) eqs. (4) are established to obtain the solutions of the given system (1).

According to the recursive scheme (3), the first column in table (12) is the same as that in table (11), while the entries in the first row of table (12) to the right of b_{11} are obtained by dividing the corresponding entries in table (11) by $b_{11}=a_{11}$; and a typical entry b_{kl} in table (12) is obtained by subtracting from the corresponding entry in table (11) the (algebraic) sum of the binary products of the elements in table (12) on the same row k to the left, with those on the same column l above, the entry b_{kl}, taken in order (first with first, second with second, etc.), when the entry is on or below the principal diagonal ($k \geqq l$), but dividing the result by the diagonal element b_{kk} on the same row k when the entry is above and to the right of the principal diagonal. It is found convenient to carry out the process by rows, from left to right, as we right in english.

In practice it is customary to add "checking" columns to the tables (11) and (12). Under the conditions assumed in the method, let us construct the following auxiliary system of equations:

$$\sum_{l=1}^{n+1} a_{kl}x_l = \alpha_k = \sum_{l=1}^{n+1} a_{kl} \text{ (by definition)} \quad (k=1,2,\ldots,m) \quad (13)$$

with the same a_{kl} ($k = 1,2,\ldots,m$; $l = 1,2,\ldots,n+1$) of the system of eqs. (1), and a new unknown x_{n+1}, and new independent terms α_k, which are taken (by definition) equal to the (algebraic) sums of the rows in table (11). This system (13) will clearly have the solution:

$$x_1 = x_2 = \ldots = x_n = x_{n+1} = 1,$$

and hence so will the following equivalent triangular system:

$$x_k + \sum_{l=k+1}^{n+1} b_{kl}x_l = \beta_k = \Big(\alpha_k - \sum_{h=1}^{k-1} b_{kh}\beta_h\Big)/b_{kk} \quad (k=1,2,\ldots,m) \quad (14)$$

formed from the system of eqs. (13) by the method. Therefore we must have:

$$1+\sum_{l=k+1}^{n+1} b_{kl} = \beta_k \,. \qquad (k=1,2,\dots,m) \qquad (15)$$

Thus if the column of α_k $(k=1,2,\dots,m)$ is added to table (11) as an extra column, and the corresponding column of β_k computed in terms of the α_k by the method (as indicated by the last members of eqs. 14) is annexed to table (12), each β_k must be equal to $\underline{\text{one}}$ plus the (algebraic) sum of all the terms in the corresponding $k^{\underline{th}}$ row of table (12) to the right of the principal diagonal. This can be (and is) used to check the process.

The method can also be used to compute the value of the determinant $\det(a_{kl})$ $(k\;\&\;l=1,2,\dots,m)$ formed with the first $\underline{m}$ columns of table (11). Because all the operations performed on the coefficients a_{kl} of the first $\underline{m}$ unknowns in eqs. (1) to obtain their coefficients b^*_{kl} $(k\;\&\;l=1,2,\dots,m)$ in eqs. (2) leave the value of the determinant unchanged, except for the divisions of the rows by $b_{11}, b_{22}, \dots, b_{mm}$, which is equivalent to the division of the value of the determinant by their product $(b_{11}b_{22}\dots b_{mm})$. Thus we have:

$$\det(a_{kl}) = b_{11}b_{22}\dots b_{mm}\cdot\det(b^*_{kl}) = b_{11}b_{22}\dots b_{mm}\,, \qquad (16)$$

since the determinant of the b^*_{kl} has a "triangular form" with $\underline{\text{ones}}$ on the principal diagonal and $\underline{\text{zeros}}$ below it, and so is clearly $=1$.

The cofactors of the elements a_{kl} $(k\;\&\;l=1,2,\dots,m)$ in the determinant (16) can also be obtained by applying the process to the following system:

$$\sum_{l=1}^{m} a_{kl}x_l = y_k \qquad (k=1,2,\dots,m), \qquad (17)$$

with indeterminates y_k used as independent terms in the second members. First this system (17) is transformed into its equivalent system in triangular form, and from the latter the x_k can easily be obtained as linear combinations of the y_l, the coefficients of which are known to be the cofactors divided by the determinant (16). Hence the cofactors can be obtained by multiplying the coefficients in these linear combinations by the determinant (16).

The cofactors, $\operatorname{cof} a_{lk}$, with a given $\underline{l}$, can also be obtained as the values of the unknowns x_k in eqs. (17) when we put $y_k=0$ $(k\neq l)$ and $y_l=\det(16)$; and, of course, this can be done by the method for $\underline{l=1,2,\dots,m}$ in a continuous process.

Let us again consider the system of $\underline{m}$ equations (1) in $\underline{n}$ unknowns, but now without any restrictions whatsoever. The trivial case in which all the a_{kl} $(k=1,2,\dots,m;\; l=1,2,\dots,n)$ are zero can *easily be disposed off; because if all the independent terms a_k $(k=1,2,\ldots,m)$ are also zero then the system is consistent, and any set of values of the unknowns constitute a solution; while if any one of the independent terms a_k is not zero, the system is inconsistent, and the eqs. (1) have no solution.*

Consider then the case in which some coefficient a_{kl} in the system of eqs. (1) is not zero. Without loss of generality we may assume (by a renumeration, if necessary) that $a_{11} \neq 0$, and with this much we can start the process and thus fill in the first row (and the first column) of table (12). Next assume that all has gone well and that we have filled up to the $(k-1)^{\underline{st}}$ row $(k>1)$. The first k *entries in the next row can be computed in succession by the first of the eqs. (3), without any trouble arising, and if $b_{kk} \neq 0$ we can compute the rest of the entries in the $k^{\underline{th}}$ row by the second of eqs. (3) and thus complete this row. But if $b_{kk}=0$ we must proceed differently.*

In the first place let us compute the following auxiliary quantities:

$$b'_{kl} = a_{kl} - \sum_{h=1}^{k-1} b_{kh} b_{hl}. \qquad (l = k+1,\ k+2, \ldots, n+1) \qquad (18)$$

(These computations will not be wasted, since they will be used later. If all these auxiliary quantities are zero, the $k^{\underline{th}}$ equation of the system (1) is linearly dependent on the first $(k-1)$ equations of the system (being then a linear combination of them) and so it can, then, be omitted. However, if all these quantities vanish except the last, namely, $b'_{k,n+1}$, we obtain the contradiction $b'_{k,n+1} = 0$, and so the system of equations (1) results inconsistent. The only situation left to consider is that in which (at least) one of the first n-k *auxiliary quantities (18) (i.e. one of the b'_{kl} with $l=k+1, k+2, \ldots, n$) does not vanish. Let the first of these not to vanish be $b'_{kl'}$. Then we can exchange the columns* k *and* l' *in tables (11) and (12), together with the corresponding unknowns x_k and $x_{l'}$ in eqs. (1) and (2). This operation will not disturb any of the values already computed in the first $(k-1)$ rows of table (12), nor the first $(k-1)$ entries in the $k^{\underline{th}}$ row, since each depends only on the corresponding entry in table (11), and on the entries in table (12) in the same column and on those in the same row to the left of the principal diagonal (the relative positions of which remain the same). In this way we obtain a non-vanishing entry in the $k^{\underline{th}}$ row on the principal diagonal, and then we can proceed to divide the previously obtained auxiliary quantities (18) by it in order to make the rest of the entries in the corresponding places of the $k^{\underline{th}}$ row of table (12), thus completing it.*

Continuing in this way, and assuming that no contradiction is obtained on the way, we can finally exhaust all the given equations before we obtain more than $\underline{n}$ equations in triangular form (omitting the ones that result linearly dependent) which are clearly linearly independent (each having at least one unknown less than the preceding one); or else we first obtain $\underline{n}$ linearly independent equations in triangular form before we exhaust all the given equations (which can happen only if $m \geq n$, of course). In the former case, the number $r (\leq n)$ of linearly independent equations in triangular form, on which all the equations of the given system depend linearly, is the rank of the system of equations, and the last $\underline{n-r}$ unknowns can be assigned arbitrarily; and then the first $\underline{r}$ unknowns can be determined from the $\underline{r}$ linearly independent equations in triangular form (by eqs. 4) in terms of the values given to the last $\underline{n-r}$ unknowns. In the latter case, the equations left over must be linearly dependent on the first $\underline{n}$ linearly independent equations already obtained, or else the originally given system of equations (1) is inconsistent. It is easy to decide which of these two is the situation; for all the equations left over will be linearly dependent on the first $\underline{n}$, if and only if, upon annexing the coefficients and independent terms of each of the equations left over, one at a time, as an $(n+1)^{\underline{st}}$ row to the table (11) (now with $m=n$) the last entry in the corresponding $(n+1)^{\underline{st}}$ (i.e. the annexed) row in table (12) obtained by the process (by eq. 18 with $l=k=n+1$) results zero for each of the equations left over; and in such a case, the given system of equations is consistent and of rank $\underline{n}$, and so no unknown can be given an arbitrary value, and hence the solution of the system of given equations is unique. This completes the discussion.

__Note__. The importance of the D-C method can be appreciated by intentionally comparing the number of operations in general necessary by this method, to solve a consistent system of $\underline{n}$ linear equations in $\underline{n}$ unknowns, with the number of operations necessary by the method of determinants. Developing the determinants by minors, the latter requires $(n+1)!-(n+1)$ additions, $(n-1)\times(n+1)!$ multiplications, and $\underline{n}$ divisions; in addition to a tremendous amount of writing. Considering that $n! = A_n (n/e)^{n+\frac{1}{2}}$, where $4 < A_n < 4.5$, so that $n! > 4(n/3)^n$, if $n>1$, it can be appreciated that the number of operations by this method is prohibiting. Even if the determinants are developed by reducing them first to triangular form, there is still need of about $\underline{n}$ times more operations than in the D-C method, besides the tremendous amount more of writing. In the D-C method there is in general necessity of $n(n-1)(2n+5)/6$ additions and this same number of multiplications, and of $n(n+1)/2$ divisions. These are the same numbers of these operations necessary by the method of successive eliminations, but the amount of writing saved is enormous.* (*Actually $e^{17/12} < A_n \leqq e^{18/12}$.)

CHAPTER VII: THE NODE METHOD.

In Ch. V, §2 (p. 149) we have outlined a general procedure to solve the complex (transformed) equations of any network of general parallel elements in the sinusoidal state. In this chapter we will establish another method of solving these equations, known as the node method, which greatly simplifies matters. This method consists essentially in making the following substitution (change of variables):

$$V_k = \sum_{n=1}^{n'_n} (k,n)\, U_n \qquad (k = 1,2,\dots,n_e) \qquad (0)$$

of the complex voltage drops V_k in terms of new (complex) quantities U_n, one for each node n $(=1,2,\dots,n'_n)$ of a complete and independent set of n'_n nodes of the network, obtained by omitting exactly one of the n_n nodes in each of the n_c components of the network (so that: $n'_n = n_n - n_c$). The U_n are called the complex node potentials, referred to the omitted nodes considered as base nodes (cf. Ch. II, §4, p. 53). Of course, U_n is the complex number corresponding to the actual sinusoidal potential (drop) $U^*(t)$ of the typical node n (to the corresponding base node in the same component as the node n).

The importance of the substitution (0) is due to the theorem expressed by eq. (1) of Ch. II, §2, by means of which we can show that the mere form of the complex voltage drops V_k given by the substitution (0) is enough to guarantee that Kirchhoff's Voltage Law shall be automatically satisfied for all the meshes of the network, no matter what values the U_n may have. It shall then only be a question of finding the U_n so as to satisfy the remaining equations of the network.

§1. THE COMPLEX CANONICAL EQUATIONS OF THE NODE METHOD.

Consider the general complex equations (6, 7, 8 of Ch. V, §2) of an arbitrary network of n_e general parallel elements in the sinusoidal state at the angular frequency $\omega>0$ connected into n_c components with n_n nodes and n_m independent meshes, namely:

$$I_k = \sum_{l=1}^{n_e} Y_{kl} V_l + I_{cs_k} + I_{vs_k} \qquad (k = 1,2,\dots,n_e) \qquad (1)$$

$$\sum_{k=1}^{n_e} (k,n) I_k = 0 \qquad (n = 1,2,\dots,n'_n = n_n - n_c) \qquad (2)$$

$$\sum_{k=1}^{n_e} [k,m]\, V_m = 0 \qquad (m = 1,2,\dots,n_m = n_e - n'_n) \qquad (3)$$

where the Y_{kl} are the admittances of and between the elements of the network at the angular frequency ω, given by eqs. (5), Ch. V, §2.

Substituting (0) into the left-hand member of (3), we obtain:

$$\sum_{k=1}^{n_e}[k,m]\left\{\sum_{n=1}^{n_n'}(k,n)U_n\right\}=\sum_{n=1}^{n_n'}\left\{\sum_{k=1}^{n_e}(k,n)\,[k,m]\right\}U_n=\sum_{n=1}^{n_n'}0\cdot U_n\equiv 0,$$

since, according to eq. (1) of Ch. II, §2, all the sums in the parentheses of the second member are zero for all m & n. Thus Kirchhoff's complex voltage law (3) is identically satisfied for all the meshes of the network.

If we now substitute (1) into (2) we get:

$$\sum_{k=1}^{n_e}\sum_{l=1}^{n_e}(k,n)Y_{kl}V_l+\sum_{k=1}^{n_e}(k,n)(I_{cs_k}+I_{vs_k})=0,\quad (n=1,2,\ldots,n_n')$$

which becomes, upon substituting the V_l as given by (0) after the summation index n is changed into another letter m (say, which will not be needed any more in connection with eqs. 4):

$$\sum_{m=1}^{n_n'}\left\{\sum_{k=1}^{n_e}\sum_{l=1}^{n_e}(k,n)Y_{kl}(l,m)\right\}U_m=-\sum_{k=1}^{n_e}(k,n)(I_{cs_k}+I_{vs_k}).\quad (n=1,2,\ldots,n_n')\qquad(4)$$

Besides these equations we have an equation (0) for each element k with a voltage source, the complex voltage rises E_k of which are (assumed) known. Thus, if the network has $n_{vs}>0$ voltage sources which, without loss of generality, we may assume to be numbered first, consecutively from 1 to $n_{vs}^{(\leq n_n')}$, then besides the n_n' equations (4), with the n_n' unknown U_m and the n_{vs} unknown I_{vs_k}, we have the n_{vs} equations:

$$\sum_{n=1}^{n_n'}(k,n)U_n=V_k=-E_k\,;\qquad (k=1,2,\ldots,n_{vs}\leq n_n')\qquad(5)$$

and, of course, the I_{vs_k} in the eqs. (4), for $k=n_{vs}+1,\ n_{vs}+2,\ldots,n_e$, may be omitted; so that they may then be rewritten as follows:

$$\sum_{m=1}^{n_n'}\left\{\sum_{k=1}^{n_e}\sum_{l=1}^{n_e}(k,n)Y_{kl}(l,m)\right\}U_m=-\sum_{k=1}^{n_e}(k,n)I_{cs_k}-\sum_{k=1}^{n_{vs}}(k,n)I_{vs_k}.\quad (n=1,2,\ldots,n_n')\qquad(6)$$

When the network has no voltage sources, i.e. $n_{vs}=0$, the complex equations of the network are the eqs. (4), only with all the I_{vs_k} omitted (the eqs. 5 also being omitted).

Equations (5) and (6), when $n_{vs}>0$, or eqs. (4) with the I_{vs_k} omitted, when $n_{vs}=0$, form a complete and independent system of $n_n'+n_{vs}$ equations with this same number of unknowns, if the admittances Y_{kl} and the complex currents I_{cs_k} through the current sources are considered as arbitrary (cf. note of Ch. III, §1, p. 64). These complex linear equations with complex coefficients of an arbitrary constant parameter (stationary) network of general parallel elements *in the sinusoidal state are called the* <u>*complex canonical node equations*</u> *of the network.*

§2. THE NODE ADMITTANCES.

The equations (4) and (6) of §1 can be given a simpler appearance by introducing the quantities $y_{mn} = y_{mn}(i\omega)$ *defined as follows:*

$$y_{mn} = \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} (k,m) Y_{kl} (l,n). \qquad (m \,\&\, n = 1,2,\ldots,n'_n) \qquad (1)$$

When the number of voltage sources $n_{vs} > 0$, *the complete and independent system of complex canonical node equations of a (stationary) sinusoidal current network of* n_e *general parallel elements with* n'_n *independent nodes can then be given in the following final form:*

$$\begin{aligned} \sum_{n=1}^{n'_n} y_{mn} U_n &= - \sum_{k=1}^{n_e} (k,m) I_{cs_k} - \sum_{k=1}^{n_{vs}} (k,m) I_{vs_k} \qquad (m = 1,2,\ldots,n'_n) \\ \sum_{n=1}^{n'_n} (k,n) U_n &= -E_k \qquad (k = 1,2,\ldots,n_{vs} \leqq n'_n) \end{aligned} \qquad (2)$$

and when $n_{vs} = 0$, *the complex canonical node equations are simply:*

$$\sum_{n=1}^{n'_n} y_{mn} U_n = - \sum_{k=1}^{n_e} (k,m) I_{cs_k} = \tilde{I}_m \text{ (say)}. \qquad (m = 1,2,\ldots,n'_n) \qquad (3)$$

The quantities y_{mn} *introduced by eqs. (1) are called the* <u>*node admittances*</u> *of the network at the angular frequency* ω *(which is the angular frequency at which the admittances of the elements are computed). It should be noticed that they depend only on the passive parts (i.e. the parameters) of the network, on the way the elements are interconnected, and on the angular frequency* ω *of the sources. The quantities* y_{mm} *are called the* <u>*self-admittances of the nodes*</u> $m = 1,2,\ldots,n'_n$, *and the quantities* y_{mn} $(m \neq n)$ *are called the* <u>*mutual-admittances of, or between, the nodes m and n*</u> $(m \,\&\, n = 1,2,\ldots,n'_n)$. *(*$y_{mm}$ *is also called the mutual admittance of the node* <u>*m*</u> *with itself.)*

If we denote the real part of y_{mn} *by* g_{mn} *and the imaginary part of* y_{mn} *by* b_{mn}, *we shall have:*

$$y_{mn} = g_{mn} + i\, b_{mn}\,. \qquad (m \,\&\, n = 1,2,\ldots,n'_n) \qquad (4)$$

The quantity g_{mn} *is called the* <u>*mutual-conductance between the nodes m & n*</u>, *and the quantity* b_{mn} *is called the* <u>*mutual-susceptance between these nodes*</u>. *When m=n, these quantities are also called the* <u>*self-conductance*</u> *and* <u>*self-susceptance*</u>, *respectively,* <u>*of the node n*</u>.

Later we will see how these node quantities can be obtained directly by inspection of the network, in terms of its parameters, and also how they can be determined experimentally.

Problem 1. Show that if $Y_{kl} = Y_{lk}$ for all k & l $(=1,2,\ldots,n_e)$ then $y_{mn} = y_{nm}$ for all m & n $(=1,2,\ldots,n_n')$; thence infer that also: $g_{mn} = g_{nm}$ and $b_{mn} = b_{nm}$ for all m & n.

Problem 2. Show that the node-admittances y_{mn} do not depend on the reference directions assigned to the elements. (Use eqs. 1.)

Problem 3. By substituting the values of the admittances Y_{kl} $(k \& l = 1,2,\ldots,n_e)$ of and between the elements, given by eqs. (5), Ch. V, §2 (p. 144), into eqs. (1), show that:

$$y_{mn} = g_{mn} + i\omega c_{mn} + \gamma_{mn}/i\omega , \qquad (m \& n = 1,2,\ldots,n_n') \qquad (5)$$

where we have put:

$$g_{mn} = \sum_{k=1}^{n_e}(k,m)G_k(k,n) = g_{nm} , \quad c_{mn} = \sum_{k=1}^{n_e}(k,m)C_k(k,n) = c_{nm} ,$$
$$\gamma_{mn} = \sum_{k=1}^{n_e}\sum_{l=1}^{n_e}(k,m)\Gamma_{kl}(l,n). \qquad (m \& n = 1, 2, \ldots, n_n') \qquad (6)$$

From this show that:

$$b_{mn} = \omega c_{mn} - \gamma_{mn}/\omega . \quad (m \& n = 1, 2, \ldots, n_n') \qquad (7)$$

The c_{mn} are called the (self- and mutual-) node capacitances, and the γ_{mn} are called the (self- and mutual-) node invertances (inverse inductances), of (or between) the nodes m & n.

Problem 4. Show that for a network without mutual-inductances (i.e. with all the $L_{kl} = 0$, if $k \neq l$; in which case all the mutual-invertances Γ_{kl} and all the mutual-admittances Y_{kl} are also 0 if $k \neq l$) the node admittances are given by:

$$y_{mn} = \sum_{k=1}^{n_e}(k,m)Y_k(k,n); \qquad (m \& n = 1,2,\ldots,n_n') \qquad (8)$$

and the node invertances are given by:

$$\gamma_{mn} = \sum_{k=1}^{n_e}(k,m)L_k^{-1}(k,n) = \sum_{k=1}^{n_e}(k,m)\Gamma_k(k,n); \qquad (9)$$

while g_{mn} and c_{mn} are (still) given by the similar (first two) eqs. (6). From these equations show that, in such a network, the mutual-admittance, conductance, capacitance, & invertance, between two distinct nodes m & n, are given by the negative sum of all the admittances, conductances, capacitances, or invertances, respectively, directly connected between the two nodes m & n; and the self-admittance, conductance, capacitance, & invertance, of the typical node n, are given by the sum of all the admittances, conductances, capacitances, or invertances, respectively, directly connected to the node n by exactly one of their terminals (all taken with a positive sign). This rule shall be found of utmost importance in practice, since it shall be *constantly needed. (HINT: When an element $\underline{k}$ has its two terminals on two distinct nodes $\underline{m}$ & $\underline{n}$, then exactly one of the factors of $(k,m)(k,n)$ is $+1$ and the other is -1, so that their product is -1.)*

Problem 5. Show that the general term $(k,m)\Gamma_{kl}(l,n)$ of the double sum (1) defining the node admittances y_{mn} shall have the same sign as Y_{kl} if and only if the elements $\underline{k}$ & $\underline{l}$ are oriented (or directed) relatively the same way to the nodes $\underline{m}$ & $\underline{n}$, respectively (i.e. if and only if both the elements are directed towards, or both are directed away from, the corresponding nodes). And the same holds for the general term $(k,m)\Gamma_{kl}(l,n)$ of the double sum defining γ_{mn} by the third of the eqs. (6).

Let us now examine eqs. (1) for the node admittances in more detail. In the first place we observe that for any given nodes $\underline{m}$ & $\underline{n}$ (not necessarily distinct) only those terms of the double sum appear for which $(k,m)\neq 0$, $(l,n)\neq 0$, and $Y_{kl}\neq 0$; this means that only those elements $\underline{k}$ inciding on the node $\underline{m}$, and only those elements $\underline{l}$ inciding on the node $\underline{n}$, need be considered. Denoting a summation over all the elements $\underline{k}$ inciding (i.e. with exactly one terminal) on the node $\underline{n}$ by $\sum\limits_{k \text{ on } n}$, equations (1) for the node admittances can be written as follows

$$y_{mn}=\sum_{k \text{ on } m}\sum_{l \text{ on } n}(k,m)Y_{kl}(l,n)=\sum_{k \text{ on } m}(k,m)\Big[\sum_{l \text{ on } n}Y_{kl}(l,n)\Big].\quad (m \& n=1,2,\dots,n'_n)\quad (10)$$

To evaluate this double sum for any given nodes $\underline{m}$ & $\underline{n}$ (not necessarily distinct) we first order the elements $\underline{k}$ around the node $\underline{m}$, and the elements $\underline{l}$ around the node $\underline{n}$, arbitrarily (e.g. clockwisely around the nodes, beginning anywhere, in the case of figures). For the first element $\underline{k'}$ (say) inciding on the node $\underline{m}$, we run through all the elements $\underline{l}$ around the node $\underline{n}$ and compute all the corresponding terms $(k',m)Y_{k'l}(l,n)$ with a $Y_{k'l}\neq 0$. Then we pass on to the next element $\underline{k''}$ (say) inciding on the node $\underline{m}$ and again we run through all the elements $\underline{l}$ around the node $\underline{n}$, computing, as we go along, all the terms $(k'',m)Y_{k''l}(l,n)$ with a $Y_{k''l}\neq 0$. We continue in this way until we have gone through all the elements $\underline{k}$ inciding on the node $\underline{m}$, each time going around all the elements $\underline{l}$ inciding on the node $\underline{n}$. Finally we add up all the terms so computed in order to obtain the admittance y_{mn} between the nodes $\underline{m}$ & $\underline{n}$.

As was said in Prob. 5, each of the terms $(k,m)Y_{kl}(l,n)$ shall be $\pm Y_{kl}$, the + sign being used if the two elements $\underline{k}$ & $\underline{l}$ are oriented relatively the same with respect to the corresponding nodes $\underline{m}$ & $\underline{n}$, and the − sign being used otherwise. This will be called the rule of signs for the admittances in the node method. It is illustrated in Fig. 1; in (a) & (b), Y_{kl} is taken with a + sign in y_{mn}, and in (c) & (d), it is taken with a - sign. (The nodes m & n need not be distinct, of course.)

$(k,m)Y_{kl}(l,n) = +Y_{kl}$ in (a) and (b). $(k,m)Y_{kl}(l,n) = -Y_{kl}$ in (c) and (d).

Fig. 1. The rule of signs for the node method.

Example 1. The self-admittance y (say) of the node 1 in the network shown in Fig. 3(a), Ch. V, §2 (p. 140), is:

$$y = Y_{22}-Y_{26}+Y_{66}-Y_{62}+Y_{55}-Y_{53}+Y_{33}-Y_{35} = Y_2+Y_3+Y_5+Y_6-2(Y_{26}+Y_{35}).$$

Problem 6. Compute the self-admittances of the nodes 0, 2, 3, & 4, of the same network shown in Fig. 3(a) of Ch. V, §2.

Example 2. The self-admittance y (say) of the node marked 0_1 in the network shown in Fig. 4(a) of Ch. V, §2 (p. 142), is:

$$y = Y_4+Y_6+Y_{62}+Y_2+Y_{26}+Y_1+Y_9.$$

Problem 7. Compute the self-admittances of the nodes 1, 2, 3, 4, & 5, of the same network shown in Fig. 4(a) of Ch. V, §2.

Example 3. Consider the transformed network of that shown in Fig. 2(a) of Ch. III, §2 (p. 73). The self-admittance of node 4 is:

$$y_{44} = Y_1-Y_{15}+Y_5-Y_{51} = G_1+i \quad C_1+(I_1+I_5 -2 I_{15}).$$

Problem 8. Compute the self-admittances of the nodes 0_1, 1, 2, 3, 5, & 6, of the same network shown in Fig. 2(a) of Ch. III, §2.

Example 4. Consider again the network of Fig. 7 (considered in Ex. 5) of Ch. V, §2 (p. 147). The self-admittance y (say) of the node 6 is:

$$y = Y_2-Y_{23}+Y_5+Y_3-Y_{32}+Y_{15}+Y_8. \qquad (Y_{15} = Y_{15,15})$$

Problem 9. Compute the self-admittances in the same network considered in Ex. 4, for the nodes 1, 2, 3, 4, & 5.

Example 5. Consider the network whose graph is shown in Fig. 8 of Ch. V, §2. The self-admittance y (say) of the node 1 is:

$$\begin{aligned} y = {} & Y_1+Y_{1,7}-Y_{1\ 16}-Y_{1,17}-Y_{1,18}-Y_{1,8}-Y_{1,2}+Y_{7,1}+Y_7-Y_{7,16}-Y_{7,17}-Y_{7,18}+ \\ & -Y_{7,8}-Y_{7,2}-Y_{16,1}-Y_{16,7}+Y_{16}+Y_{16,17}+Y_{16,18}+Y_{16,8}+Y_{16,2}-Y_{17,1}+ \\ & -Y_{17,7}+Y_{17,16}+Y_{17}+Y_{17,18}+Y_{17,8}+Y_{17,2}-Y_{18,1}-Y_{18,7}+Y_{18,16}+Y_{18,17}+ \\ & +Y_{18}+Y_{18,8}+Y_{18,2}-Y_{8,1}-Y_{8,7}+Y_{8,16}+Y_{8,17}+Y_{8,18}+Y_8+Y_{8,2}-Y_{2,1}+ \\ & -Y_{2,7}+Y_{2,16}+Y_{2,17}+Y_{2,18}+Y_{2,8}+Y_2. \end{aligned}$$

Problem 10. Compute the self-admittance of the node 2 in the same network considered in the preceding example 5.

Example 6. The mutual-admittance y_M (say) between the nodes 0 and 1, of the network shown in Fig. 3(a) of Ch. V, §2 (p. 140), is:

$$y_M = Y_{46} - Y_5 + Y_{53} + Y_{35} - Y_3.$$

Problem 11. For the same network considered in the preceding example 6, compute the mutual-admittances between the following pairs of nodes: 1 & 2, 1 & 4, 1 & 3, 2 & 3, 2 & 4, and 3 & 4.

Example 7. The mutual-admittance y_M (say) between the nodes 0_1 and 3, of the network shown in Fig. 4(a) of Ch. V, §2 (p. 142), is:

$$y_M = -Y_{62} + Y_{67} - Y_2 + Y_{27} - Y_1 + Y_{95};$$

and the mutual-admittance between the nodes 1 and 3, y'_M (say), is:

$$y'_M = Y_{72} - Y_7;$$

while that between 1 and 4 is: $y_{14} = -Y_{4,10}$; whereas: $y_{15} = 0$.

Problem 12. Compute the mutual-admittances between the following pairs of nodes: 0_1 & 2, 2 & 5, 2 & 3, 2 & 4, and 1 & 5, of the same network considered in the preceding example 7.

Example 8. Consider the transform of the network shown in Fig. 2(a) of Ch. III, §2 (p. 73). The mutual-admittance between the nodes 0_1 and 4 is:

$$y_M \text{ (say)} = Y_{21} - Y_1 + Y_{15} = -G_1 - i\omega C_1 + (\Gamma_{12} - \Gamma_1 + \Gamma_{15})/i\omega\ ;$$

and that between the nodes 3 and 4 is: $y_{34} = -Y_5 + Y_{51} = (\Gamma_{51} - \Gamma_5)/i\omega$;
and that between the nodes 1 and 4 is: $y_{14} = -Y_{21} + Y_{25} = (\Gamma_{25} - \Gamma_{12})/i\omega$;
while: $y_{24} = 0$, $y_{23} = -Y_4 = -G_4$, $y_{25} = 0$, $y_{45} = -Y_{56}$, $y_{35} = Y_{56}$, $y_{26} = -Y_{37}$.

Problem 13. Compute the mutual-admittances between the following pairs of nodes: 0_1 & 1, 0_1 & 2, 0_1 & 3, 0_1 & 5, 5 & 6, 6 & 0_3, and 4 & 6, of the same network considered in the preceding example 8.

Example 9 Consider the network whose graph is shown in Fig. 7 of Ch. V, §2 (which was considered in the corresponding example of p. 147). The mutual-admittance between the nodes 4 and 6 is:

$$y_{46} = -Y_{12} + Y_{13} - Y_5 + Y_{42} - Y_{43}.$$

Problem 14. Compute the mutual-admittances between the following pairs of nodes: 4 & 5, 5 & 6, 5 & 7, 5 & 1, 5 & 2, and 5 & 3, of the same network considered in the preceding example 9.

Example 10. Consider the network whose graph is shown in Fig. 8 of Ch. V, §2. The mutual-admittance y_M between the nodes 1 and 5 is:

$$\begin{aligned} y_M = {} & Y_{1,6}-Y_{1,12}+Y_{1,16}+Y_{1,19}+Y_{1,21}+Y_{1,11}+Y_{1,5}-Y_{7,6}-Y_{7,12}+Y_{7,16}+ \\ & +Y_{7,19}+Y_{7,21}+Y_{7,11}+Y_{7,5}+Y_{16,6}+Y_{16,12}-Y_{16}-Y_{16,19}-Y_{16,21}-Y_{16,11}+ \\ & -Y_{16,5}+Y_{17,6}+Y_{17,12}-Y_{17,16}-Y_{17,19}-Y_{17,21}-Y_{17,11}-Y_{17,5}+Y_{18,6}+ \\ & +Y_{18,12}-Y_{18,16}-Y_{18,19}-Y_{18,21}-Y_{18,11}-Y_{18,5}+Y_{8,6}+Y_{8,12}-Y_{8,16}+ \\ & -Y_{8,19}-Y_{8,21}-Y_{8,11}-Y_{8,5}+Y_{2,6}+Y_{2,12}-Y_{2,16}-Y_{2,19}-Y_{2,21}-Y_{2,11}-Y_{2,5}. \end{aligned}$$

Problem 15. Compute the mutual-admittance between the nodes 1 and 2 in the same network considered in the preceding example 10.

The node admittances y_{mn} obtained in terms of the admittances Y_{kl} of, and between, the elements of a network can be expressed in more detail in terms of the conductances, capacitances, and (self-amd mutual-) invertances of the basic elements of the network by replacing (in accordance with eqs. 5 of Ch. V, §2) each self-admittance $Y_k = Y_{kk}$ by $G_k + i\omega C_k + \Gamma_k/i\omega$, and each mutual-admittance Y_{kl} $(k \neq l)$ by $\Gamma_{kl}/i\omega$. From this we infer that any conductance or capacitance appearing in a node admittance y_{mn} between two distinct nodes can only be due to a self-admittance of an element connected directly between them, and conversely; whereas all other terms in y_{mn} shall be of the form $\Gamma_{kl}/i\omega$. And any conductance or capacitance appearing in a node admittance y_{nn} of a node _n_ can only be due to a self-admittance of an element connected to the node _n_ by exactly one of its terminals, and conversely; whereas all the other terms of y_{nn} shall be of the form $\Gamma_{kl}/i\omega$.

The double summation for the node admittances y_{mn} given by eqs. (1) (or eqs. 10), can also be obtained by first summing the terms with $k=l$ and then adding the sum of the terms with $k \neq l$. This may be expressed thus:

$$y_{mn} = \sum_{k \text{ on } m\&n} (k,m) Y_{kk} (k,n) + \sum_{\substack{k \text{ on } m \\ l \text{ on } n \,\&\, l \neq k}} (k,m) Y_{kl} (l,n). \qquad (11)$$
$$(m \,\&\, n = 1, 2, \ldots, n'_n)$$

The last sum (for the terms with $k \neq l$) may in turn be split into the sum of those terms corresponding to distinct elements _k_ & _l_ (both) inciding on both the nodes _m_ & _n_, and the sum of those terms corresponding to distinct elements _k_ & _l_ not (both) inciding on both the nodes _m_ & _n_. Thus eqs. (11) may be expressed as follows (putting $Y_k = Y_{kk}$):

$$y_{mn} = \sum_{k \text{ on } m\&n} (k,m) Y_k (k,n) + \sum_{\substack{k\&l \text{ on } m\&n \\ k \neq l}} (k,m) Y_{kl} (l,n) + \sum_{\substack{k\&l \text{ not on } m\&n \\ k \neq l}} (k,m) Y_{kl} (l,n) \qquad (12)$$
$$(m \,\&\, n = 1, 2, \ldots, n'_n)$$

All the terms with $k=l$ correspond to elements common to both the nodes _m_ & _n_, and they are all of the form $(k,m) Y_{kk} (k,n) = \pm Y_{kk} = \pm Y_k$, the + sign always being used when $m=n$ (since then $(k,m)=(k,n)$ and so $(k,m)(k,n) = (\pm 1)^2 = 1$), and the − sign always being used when $m \neq n$

(since then $(k,m)=-(k,n)$, as can easily be checked by considering all the possible cases, and so $(k,m)(k,n) = -1)$.

All the terms with $k\neq l$ corresponding to pairs of distinct elements (both) inciding on both the nodes m & n always appear in pairs: $(k,m)Y_{kl}(l,n) + (l,m)Y_{lk}(k,n)$. Now since $L_{kl}=L_{lk}$ so that $\Gamma_{kl}=\Gamma_{lk}$ (for constant parameter networks, as is always assumed to be true in alternating current networks), we have $Y_{kl}=Y_{lk}$. Therefore the above pair of terms can be written $[(k,m)(l,n)+(l,m)(k,n)]Y_{kl} = \pm 2Y_{kl}$ (since we shall always have: $(k,m)(l,n)=(l,m)(k,n)$ $(=\pm 1)$, as can be checked by considering all possible cases). When $m=n$, the + sign is used when the two elements k & l are directed towards, or both are directed away from, the node n, and the - sign is used otherwise. When $m\neq n$, the - sign is used when the two elements k & l are directed towards, or both are directed away from, the node m (or n), and the + sign is used otherwise (in accordance with the rule of signs of Fig. 1).

All the terms with $k\neq l$ corresponding to pairs of distinct elements, one inciding on each node but not (both) inciding on both the nodes m & n, are all of the form $(k,m)Y_{kl}(l,n) = \pm Y_{kl}$ and can only appear singly (and not in pairs) and this, in fact, only when $m\neq n$, each with a sign given by the rule of signs exhibited in Fig. 1. When $m=n$, these terms are not present.

We can summarize all the above into the following practical rule for the computation, by inspection, of the node admittances of a network. First we shall consider the case $m=n$ of a self-admittance of a node, and then the case $m\neq n$ of a mutual-admittance between two distinct nodes.

1º). The self-admittance y_{nn} of the node n is the sum of all the self-admittances of all the elements inciding on this node (i.e. with exactly one terminal on this node), all taken with a + sign, plus twice the (algebraic) sum of all the (non-zero) mutual-admittances between pairs of distinct elements inciding on this node, taken with a + sign when the two elements are both directed towards, or both away from, the node n, and with a - sign otherwise.

By substituting the admittances of the elements according to eqs. (5), Ch. V, §2, this rule can be stated in more detail as follows: y_{nn} is the (arithmetical) sum of all the conductances inciding on the node n (i.e. with exactly one terminal on the node n, all taken with a + sign), plus $i\omega$ multiplied by the (arithmetical) sum of all the capacitances inciding on the node n (all taken with a + sign), plus $1/i\omega$ multiplied by the (arithmetical) sum of all the self-invertances of the (coils of the) elements inciding on the node n (all taken with a + sign) in addition to twice the (algebraic) sum of all the mutual-invertances between pairs of distinct (coils of) elements inciding on the node n (taken with a + sign when both the elements are directed towards, or both are directed away from, the node n, and with a - sign otherwise).

When Γ-polarity marks for the mutual-invertances between the coils of the elements are given, the final signs with which the absolute values of the mutual-invertances are present in the self-admittance y_{nn} of the node n shall depend only on the relative positions of the Γ-polarity marks with respect to the node n, and not on the reference directions assigned to the elements, as can be checked by considering all (the finite number of) possible cases in accordance with the rule of signs for the mutual-invertances given in Ch. I, §9. The sign with which the absolute value $|\Gamma_{kl}|$ of a mutual-invertance (present in the self-admittance y_{nn} of the node n) appears shall be a + sign if the Γ-polarity marks are both on the same sides of the coils k & l as the node n or if both are on the opposite sides; otherwise it shall be a - sign. (See Fig. 2)

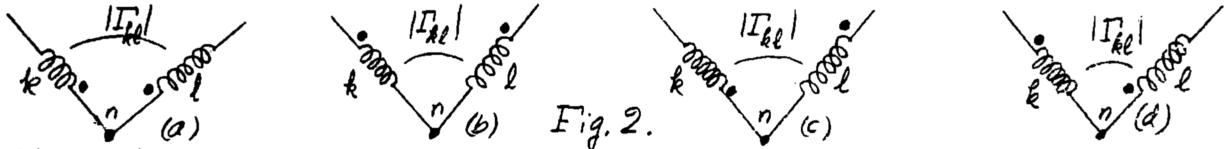


Fig. 2.

In (a) and (b), $|\Gamma_{kl}|$ appears in y_{nn} with a + sign. In (c) and (d), $|\Gamma_{kl}|$ appears with a - sign in y_{nn}.

2º). The mutual-admittance y_{mn} between two distinct nodes m and n is the negative sum of all the self-admittances of all the elements inciding at the same time on both the nodes m & n (i.e. with exactly one terminal on each of the nodes m & n, all taken with a - sign), plus twice the (algebraic) sum of all the (non-zero) mutual- admittances between pairs of distinct elements inciding on both the nodes m & n (i.e. having terminals on both nodes, each taken with a — sign when the elements of the pair are both directed towards, or both are directed away from, the node m (or n), and with a + sign otherwise), plus the (algebraic) sum of all the (non-zero) mutual-admittances between pairs of distinct elements, one inciding on each of the nodes m & n, but not (both) having terminals on both nodes (each taken with a sign given by the rule of signs for the mutual-admittances exhibited in Fig. 1).

By substituting the values of the admittances of the elements according to eqs. (5), Ch. V, §2; we get the following more detailed statement of this rule. The mutual-admittance y_{mn} between the distinct nodes m & n is equal to the negative of the (arithmetical) sum *of all the conductances (of the resistors) with one terminal on each of the nodes m & n, plus $i\omega$ multiplied by the negative of the (arithmetical) sum of all the capacitances (of the condensers) with terminals on both the nodes m & n, plus $1/i\omega$ multiplied by the negative of the (arithmetical) sum of all the self-invertances (of the coils) with terminals on both the nodes m & n, plus $1/i\omega$ multiplied by twice the (algebraic) sum of all the mutual-invertances between pairs of distinct coils both of which have terminals on both the nodes m & n (taken with a − sign when the coils of a pair are both directed towards, or both are directed away from, the node m (or n), and with a + sign otherwise), plus $1/i\omega$ multiplied by the (algebraic) sum of all the mutual-invertances between pairs of distinct coils, one inciding on one of the nodes m or n and the other inciding on the other node, but not both inciding on both the nodes m & n (each mutual-invertance being taken with a sign given by the rule of signs for the mutual-invertances exhibited in Fig. 1).*

When Γ-polarity marks for the mutual-invertances Γ_{kl} between the coils are given, the final sign with which the absolute value $|\Gamma_{kl}|$ appears in the mutual-admittance y_{mn} in which it is present shall depend only on the relative positions of the Γ-polarity marks with respect to the nodes m & n and not on the reference directions assigned to the elements. If the Γ-polarity marks are both on the same sides of the coils as the two nodes m & n, or if both are on the opposite sides, then $|\Gamma_{kl}|$ shall appear in y_{mn} with a + sign; otherwise it shall appear with a − sign. This is illustrated in Fig. 3.

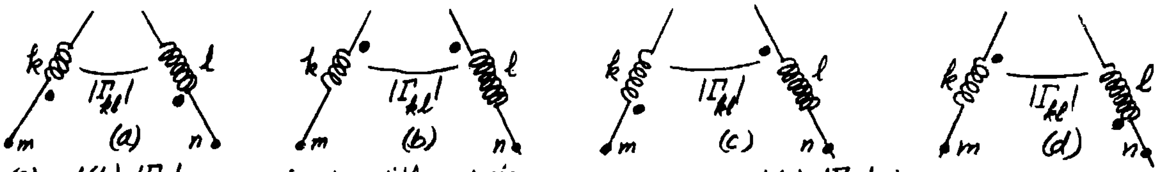


Fig. 3.

In (a) and (b), $|\Gamma_{kl}|$ appears in y_{mn} with a + sign. *In (c) and (d), $|\Gamma_{kl}|$ appears in y_{mn} with a − sign.*

Example 11. The self-admittance of the node 1 in the network shown in Fig. 3(a), Ch. V, §2 (p. 140), is:

$$y_{11} = G_2+G_3+i\omega(C_2+C_3) + (\Gamma_2+\Gamma_3+\Gamma_5+\Gamma_6-2\Gamma_{26}-2\Gamma_{35})/i\omega .$$

This can be checked with the result of example 1.

Example 12. The self-admittance y (say) of the node O_1 in the network shown in Fig. 4(a) of Ch. V, §2 (p. 142), is:

$$y = G_4+G_6+G_9+i\omega(C_1+C_2+C_4+C_9) + (\Gamma_1+\Gamma_2+\Gamma_4+\Gamma_6+\Gamma_9+2\Gamma_{26})/i\omega,$$

as can be checked with the result of example 2.

Example 13. The self-admittance y (say) of the upper node in the network shown in Fig. 5(c) of Ch. V, §2 (p. 144), is:

$$y = G_1 + G_2 + i\omega(C_1 + C_2) + (\Gamma_1 + \Gamma_2 + 2\,\Gamma_{12})/i\omega\,.$$

Example 14. The self-admittance of the node 1 in the network shown in Fig. 6, Ch. V, §2 (p. 146) is:

$$y_{11} = G_1 + i\omega C_1 + \Gamma_1/i\omega\,.$$

Example 15. The mutual-addmittance y_M (say) between the nodes and 1, of the network shown in Fig. 3 (a), Ch. V, §2, (p. 140) is:

$$y_M = -G_3 - i\omega C_3 - (\Gamma_3 + \Gamma_5)/i\omega + 2\,\Gamma_{35}/i\omega + \Gamma_{46}/i\omega,$$

as can be checked with the result of example 6.

Example 16. The mutual-admittance y_M (say) between the nodes $_1$ and 3, of the network shown in Fig. 4(a), Ch. V, §2, (p. 142) is:

$$y_M = -i\omega(C_1+C_2) -(\Gamma_1+\Gamma_2)/i\omega - \Gamma_{26}/i\omega +\Gamma_{67}/i\omega +\Gamma_{27}/i\omega +\Gamma_{59}/i\omega\,;$$

and the mutual-admittance between the nodes 1 and 3, y'_M (say), is:

$$y'_M = -G_7 - \Gamma_7/i\omega +\Gamma_{27}/i\omega\,.$$

These can be checked with the results of example 7.

Example 17. The mutual-admittance y_M (say) between the upper node of the elements 1 & 2 and the right-hand node of the element 3, in the network shown in Fig. 5(c) of Ch. V, §2 (p. 144), is:

$$y_M = (\Gamma_{13} - \Gamma_{23})/i\omega\,.$$

Example 18. The mutual-admittance y_M (say) between the nodes $_1$ and 5 of the network shown in Fig. 6, Ch. V, §2 (p. 146), is:

$$y_M = (\Gamma_{14} - \Gamma_{25})/i\omega\,.$$

Problem 16. Compute directly, in terms of conductances, capacitances, and (self- and mutual-) invertances, the self- and mutual-admittances asked for in the problem 6, 7, 8, 11, 12, and 13 (above).

§3. THE COMPLEX NODAL QUANTITIES AND THEIR INTERPRETATION.

It is easy to give an interpretation of the complex potentials U_n $(n = 1,2,\dots,n'_n)$ introduced by the eqs. (0) of the introduction to this chapter. For this purpose, let us first consider an element k with its two terminals on the distinct nodes n' and n" (oriented from n' to n") as shown in Fig. 1, and let 0 be the (arbitrarily chosen)

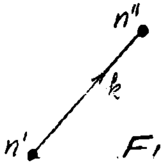


Fig. 1.

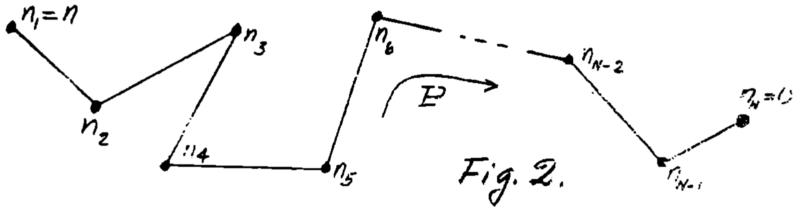


Fig. 2.

base node on the same component as the element k. Then, putting $U_0=0$, we always have:

$$V_k = \sum_{n=1}^{n'_n}(k,n)U_n = (k,n')U_{n'} + (k,n'')U_{n''} = U_{n'} - U_{n''}, \quad ($$

so that V_k is equal to the potential at its tail-terminal minus the potential at its head-terminal. Next consider any node n on the same component as the base node 0, and let P be a (simple) path through elements of the network joining n to 0. Let $n_1(=n)\, n_2\, n_3 \ldots n_N(=0)$ be the sequence of nodes along the path P from n to 0. If an element of the path, connected between the nodes n_k and n_{k+1}, is directed in the sense of the path from n to 0, the complex voltage drop through it in the reference direction assigned to it is $U_{n_k} - U_{n_{k+1}}$, which is the same as the complex voltage drop through it in the sense of the path from n to 0; and if the element is directed in the opposite sense, the complex voltage drop through it in its assigned reference direction is $U_{n_{k+1}} - U_{n_k}$, and the complex voltage drop through it in the sense of the path from n to 0 is then $-(U_{n_{k+1}} - U_{n_k}) = U_{n_k} - U_{n_{k+1}}$. Consequently, no matter how the elements of the path P are directed, the total complex voltage drop along the path from n to 0 is:

$$V_P = V_{n0} = (U_{n_1} - U_{n_2}) + (U_{n_2} - U_{n_3}) + \ldots + (U_{n_{N-1}} - U_{n_N}) = U_n - U_0 = U_n.$$

Thus the complex potential U_n is the complex voltage drop from the node n to the base node 0 in the same component of the network, along a path through elements of the network from the node n to the node 0.

According to Ch. II, §4, p. 53, we already knew that if the U_n were the complex voltage drops from the nodes n to the base nodes in the corresponding components, then we had:

$$V_k = \sum_n (k,n)U_n; \qquad (k = 1,2,\ldots,n_e) \qquad (2)$$

but now we have shown that if new (as yet uninterpreted) quantities U_n are introduced by these equations, then the new quantities are the complex voltage drops from the nodes n to the base nodes in the corresponding components of the network, to be called then the complex potentials of the nodes n $(=1,2,\ldots,n'_n)$ with respect to the corresponding base nodes. Based on this, all the equations (1) or (2) can be established directly by inspection, for any given network, as soon as all the reference directions are assigned to the elements of the network, in any way whatsoever, and as soon as a complete and independent set of n'_n nodes is arbitrarily chosen by omitting exactly one of the n_n nodes in each of the n_c components (=separate connected parts) of the network

It is also easy to give an interpretation of the right-hand members of eqs. (3) and of the first of eqs. (2), of §2. For I_{cs_k} and I_{vs_k} are the complex currents through the current and voltage sources of the element $\underline{k}$, in the reference direction assigned to the element $\underline{k}$, and so $(k,n)I_{cs_k}$ and $(k,n)I_{vs_k}$ are the complex currents leaving the node $\underline{n}$ through the current and voltage sources of the element $\underline{k}$ (as can be checked by considering all the possible cases; see Prob. 2, Ch. II, §3, p. 43); so that $-(k,n)I_{cs_k}$ and $-(k,n)I_{vs_k}$ are the complex currents entering the node $\underline{n}$ through the current and voltage sources of the element $\underline{k}$, and consequently the right-hand members of the eqs. (3) and of the first of eqs. (2), of §2, are the (algebraic) sums of all the complex currents entering the nodes $\underline{n}$ through the (current and voltage) sources connected directly (by exactly one terminal) to these nodes.

Thus the right-hand members of the complex canonical nodal equations of any given (stationary) network can be established directly by inspection. Moreorver, the left-hand members of these equations are also easy to establish systematically. Thus, for the first of the eqs. (2) and the eqs. (3), of §2, one needs only to write down, systematically, expressions (sums) such as: $y_{m1}U_1 + y_{m2}U_2 + \dots + y_{mn'_n}U_{n'_n}$, with as many terms n'_n as there are independent nodes in the network, one for each of the independent nodes $\underline{n}$ $(=1,2,\dots,n'_n)$. Furthermore, the complex coefficients y_{mn} $(m \,\&\, n = 1,2,\dots,n'_n)$ in these expressions may readily be obtained by inspection of the given network, by the methods and rules given in the preceding section 2. Thus all the complex canonical nodal equations of any given (stationary) network can be completely established directly from the network.

Essentially, the first of eqs. (2), and the eqs. (3), of §2, express Kirchhoff's complex current law for the nodes $\underline{m}$ $(=1,2,\dots,n'_n)$ of the network. And since the right-hand members of these equations are the (algebraic) sums of the complex currents entering the nodes $\underline{m}$ through the (current and voltage) sources (i.e. the active parts of the elements) directly connected to them, the left-hand members of these equations, namely: $\sum_n y_{mn}U_n$, must be the (algebraic) sums of the complex currents leaving these nodes $\underline{m}$ through the other (i.e. the passive) parts of the elements directly connected to them (i.e. through the resistors, condensers, and coils, directly connected to them). The contribution of the complex nodal potential U_n $(n=1,\dots,n'_n)$ to the total complex current leaving the node $\underline{m}$ through the passive parts of the elements connected to it is then $y_{mn}U_n$. Consequently, the mutual-admittance y_{mn} between the nodes *m* & *n* may be interpreted as the (algebraic) sum of all the complex currents leaving the node *m* through the passive parts of the elements directly connected to it (by exactly one terminal), due to a unit complex potential $1\underline{/0}^\circ = 1+0i$ of the node *n*. In particular, when $m=n$, the self-admittance y_{nn} of the node *n* is the (algebraic) sum of all the complex currents leaving the node *n* through the passive parts of the elements directly connected to it (by exactly one terminal), due to a unit complex potential, $U_n = 1\underline{/0}^\circ = 1+0i = 1$, of this node.

These remarks give us another method of obtaining the nodal admittances y_{mn} $(m \,\&\, n = 1,2,\ldots,n'_n)$ experimentally or by inspection. It consists in imagining (or producing) a unit complex voltage drop from the node *n* to the base node in the same component and then determining by inspection (or experimentally) the total complex current it causes to leave the node *m* through the passive parts of the elements connected directly (by exactly one terminal) to the node *m*; and this is precisely y_{mn}. This is in fact the method found (and used) in the older literature.

The terms $y_{mn}U_n$ in the left-hand members of the complex canonical node equations expressing Kirchhoff's complex current law for the nodes *m* will be called *yU-currents*. The term $y_{nn}U_n$ will be called the *self-yU-current* leaving the node *n* (through the passive elements), due to its own complex potential U_n; and the term $y_{mn}U_n$ $(m \neq n)$ will be called the *mutual-yU-current* leaving the node *m* (through the passive elements), due to the complex potential U_n of the node *n*. We can then state that the total complex current leaving the typical node *n* (through the passive elements) is equal to the sum of all the self- and mutual-yU-currents leaving this node (through the passive elements), due to all the complex nodal potentials; and this, of course, is also equal to the total complex current entering the node *n* through the active elements (i.e. through the current and voltage sources) directly connected to it by exactly one terminal.

Problem 1. Assume we are given an arbitrary a-c network. Suppose we removed (or disconnect) all the sources, but otherwise leave the rest of the (basic) elements as they were. This, of course, shall not affect the nodal impedances, because they do not depend on the sources (except for their frequencies). Now short-circuit (i.e. identify) all the nodes of the network with the corresponding base nodes, *except the node $\underline{n}$ to which a voltage source is connected from the corresponding base node in the same component, of a value $\sqrt{2}\sin\omega t \leftrightarrow 1\angle 0°$. Show that the self-admittance y_{nn} of the node $\underline{n}$* (at the angular frequency ω) *is equal to the complex current through the voltage source, flowing from the base node, on the same component as $\underline{n}$, to the node $\underline{n}$. Show also that the mutual-admittance y_{mn} between the nodes $\underline{m}$ & $\underline{n}$* (at the angular frequency ω) *is equal to the complex current entering the node $\underline{m}$ through the short-circuit connecting it to the corresponding base node in the same component. (Hint: Show that these complex currents are equal to the total complex currents leaving the nodes $\underline{n}$ and $\underline{m}$, respectively, through the passive (basic) elements connected directly to them.)*

<u>Example 1</u>. Consider the network shown in Fig. 3(a), of the example 2, Ch. V, §2 (p. 140). This network has $n_n' = 4$ independent nodes and $n_{vs} = 1$ voltage source. Using the same complete and independent set of nodes used then, the complex canonical node equations of the network are the following:

$$
\begin{aligned}
&y_{11}U_1 + y_{12}U_2 + y_{13}U_3 + y_{14}U_4 = I_{cs_2} + I_{cs_3} + I_{vs} &&\text{(node 1)}\\
&y_{21}U_1 + y_{22}U_2 + y_{23}U_3 + y_{24}U_4 = -I_{cs_2} &&\text{(node 2)}\\
&y_{31}U_1 + y_{32}U_2 + y_{33}U_4 + y_{34}U_4 = 0 &&\text{(node 3)}\\
&y_{41}U_1 + y_{42}U_2 + y_{43}U_3 + y_{44}U_4 = -I_{cs_2} &&\text{(node 4)}
\end{aligned}
$$

$$U_1 = E \text{ (the known voltage rise in the vs)}$$

where:

$$
\begin{aligned}
y_{11} &= Y_2 + Y_3 + Y_5 + Y_6 - 2(Y_{26} + Y_{35})\\
y_{22} &= Y_1 + Y_4 + Y_7\\
y_{33} &= Y_7 + Y_8\\
y_{44} &= Y_2 + Y_6 + Y_8 - 2Y_{26}\\
y_{12} &= -Y_{46} = y_{21}\\
y_{13} &= 0 = y_{31}\\
y_{14} &= -Y_2 - Y_6 + 2Y_{26} = y_{41}\\
y_{23} &= -Y_7 = y_{32}\\
y_{24} &= Y_{46} = y_{42}\\
y_{34} &= -Y_8 = y_{43}.
\end{aligned}
$$

These results can be checked with those of Problem 6, Ch. V, §2.

<u>Example 2</u>. For the network considered in example 3, shown in Fig. 4(a), of Ch. V, §2 (p. 142), the complex canonical node equations with respect to the same complete & independent set of nodes used then are the following:

$$y_{11}U_1 + y_{12}U_2 + y_{13}U_3 + y_{14}U_4 + y_{15}U_5 = -I_{cs_3} \quad (\text{with } U_3 = -E = \text{known})$$
$$y_{21}U_1 + y_{22}U_2 + y_{23}U_3 + y_{24}U_4 + y_{25}U_5 = I_{cs_3}$$
$$y_{31}U_1 + y_{32}U_2 + y_{33}U_3 + y_{34}U_4 + y_{35}U_5 = I_{cs_1} - I_{vs}$$
$$y_{41}U_1 + y_{42}U_2 + y_{43}U_3 + y_{44}U_4 + y_{45}U_5 = 0$$
$$y_{51}U_1 + y_{52}U_2 + y_{53}U_3 + y_{54}U_4 + y_{55}U_5 = 0$$

where:
$$y_{11} = Y_3 + Y_4 + Y_7$$
$$y_{22} = Y_3 + Y_6 + Y_5 + Y_9 + 2Y_{59}$$
$$y_{33} = Y_1 + Y_2 + Y_7 + Y_5 - 2Y_{27}$$
$$y_{44} = Y_{10}$$
$$y_{55} = Y_8$$
$$y_{12} = -Y_3 + Y_{67} = y_{21}$$
$$y_{13} = -Y_7 + Y_{27} = y_{31}$$
$$y_{14} = -Y_{4,10} = y_{41}$$
$$y_{15} = 0 = y_{51}$$
$$y_{23} = -Y_5 + Y_{26} - Y_{67} - Y_{59} = y_{32}$$
$$y_{24} = 0 = y_{42}$$
$$y_{25} = Y_{58} + Y_{89} = y_{52}$$
$$y_{34} = 0 = y_{43}$$
$$y_{35} = -Y_{58} = y_{53}$$
$$y_{45} = 0 = y_{54}.$$

These results can be checked with those of Prob. 6, Ch. V, §2.

Example 3. The complex nodal equations for the network shown in Fig. 5(c), Ch. V, §2, assumed to be in the sinusoidal state at the angular frequency $\omega > 0$, are:

$$[G_1+G_2+i\omega(C_1+C_2)+(\Gamma_1+\Gamma_2+2|\Gamma_{12}|)/i\omega]E + (|\Gamma_{12}|-|\Gamma_{23}|)U_2 = I_{cs} + I_{vs}$$
$$(|\Gamma_{12}|-|\Gamma_{23}|)E + (G_3 + i\omega C_3 + \Gamma_3/i\omega)U_2 = 0.$$

Example 4. Consider the network treated in Ex. 4, shown in Fig. 6, of Ch. V, §2. The complex canonical node equations are:

$$\sum_{n=1}^{5} y_{mn}U_n = \tilde{N}_m \qquad (m = 1,2,3,4,5)$$

where $\tilde{N}_1 = -I_{cs_1}$, $\tilde{N}_2 = -I_{cs_2}$, $\tilde{N}_3 = -I_{vs}$, $\tilde{N}_4 = \tilde{N}_5 = 0$, $U_3 = -E$,

and:
$$y_{11} = G_1 + i\omega C_1 + \Gamma_1/i\omega$$
$$y_{22} = G_2 + i\omega C_2 + \Gamma_2/i\omega$$
$$y_{33} = G_3 + i\omega C_3 + \Gamma_3/i\omega$$
$$y_{44} = G_4+G_6 + i\omega(C_4+C_6) + (\Gamma_4+\Gamma_6)/i\omega$$
$$y_{55} = G_4+G_5 + i\omega(C_4+C_5) + (\Gamma_4+\Gamma_6)/i\omega$$
$$y_{12} = y_{21} = y_{13} = y_{31} = y_{23} = y_{32} = y_{24} = y_{42} = y_{35} = y_{53} = 0$$

$$y_{14} = y_{41} = \Gamma_{14}/i\omega\,, \qquad y_{15} = y_{51} = -\Gamma_{14}/i\omega$$
$$y_{25} = y_{52} = \Gamma_{25}/i\omega\,, \qquad y_{34} = y_{43} = -\Gamma_{36}/i\omega$$
$$y_{45} = y_{54} = -(G_4 + i\omega C_4 + \Gamma_4/i\omega).$$

These results can be checked with the help of the equations given in the example 4 of Ch. V, §2, by eliminating the complex currents I_k $(k = 1,\ldots,6)$, upon substituting their expressions given by the first six equations into the equations expressing Kirchhoff's complex current laws for the node 1 to 5, and then placing $V_4 = -V_5 - V_6$, and making use of the fact that: $V_1 = U_1$, $V_2 = U_2$, $V_3 = U_3$, $V_5 = U_5$, $V_6 = -U_4$.

Problem 2. Establish the complex canonical node equations for the networks shown in figures 5, 7, 8, & 9, of Ch. V, §2.

§4. THE SOLUTION OF THE CANONICAL NODAL EQUATIONS.

When the number n_{vs} of voltage sources in a given network of n_e general parallel elements is zero, the corresponding complex canonical node equations (3) of §2, namely,

$$\sum_{n=1}^{n_n'} y_{mn} U_n = -\sum_{k=1}^{n_e} (k,m)\, I_{cs_k} = \tilde{I}_m \text{ (say)}, \qquad (m = 1,2,\ldots,n_n') \quad (1)$$

can be solved for the complex nodal potentials in the following form:

$$U_n = \sum_{m=1}^{n_n'} z_{nm} \tilde{I}_m = \sum_{m=1}^{n_n'} \tilde{I}_m \operatorname{cof} y_{mn}/\det = -\sum_{k=1}^{n_e} \sum_{m=1}^{n_n'} I_{cs_k} (k,m) \operatorname{cof} y_{mn}/\det, \quad (2)$$
$$(n = 1,2,\cdots,n_n')$$

when the determinant, $\det = \det(y_{mn})$, of the nodal admittances is not zero, where we have put: $z_{mn} = \operatorname{cof} y_{nm}/\det$, for m & $n = 1,2,\ldots,n_n'$.

And when $n_{vs} > 0$, the corresponding complex canonical node equations (2) of §2, namely (assuming the elements with voltage sources to be numbered first, $1,2,\ldots,n_{vs}$, and the others last, from $n_{vs}+1$ to n_e):

$$\sum_{n=1}^{n_n'} y_{mn} U_n + \sum_{k=1}^{n_{vs}} (k,m) I_{vs_k} = -\sum_{k=1}^{n_e} (k,m) I_{cs_k} = \tilde{I}_m \qquad (m = 1,2,\ldots,n_n')$$
$$\sum_{n=1}^{n_n'} (k,n) U_n = -E_k \qquad (k = 1,2,\ldots,n_{vs} \leqq n_n') \qquad (3)$$

can be solved for the complex nodal potentials and the complex currents through the voltage sources in the following way.

Let us introduce the following quantities:

$$\det = \det(y_{mn}) \text{ assumed } \neq 0 \qquad (\text{with } m \,\&\, n = 1,2,\ldots,n_n')$$
$$z_{nm} = \operatorname{cof} y_{mn}/\det \qquad (n \,\&\, m = 1,2,\ldots,n_n')$$
$$\tilde{z}_{kl} = \sum_{m=1}^{n_n'} \sum_{n=1}^{n_n'} (k,n) z_{nm} (l,m) \qquad (k \,\&\, l = 1,2,\ldots,n_{vs}) \qquad (3')$$
$$\tilde{E}_k = \sum_{m=1}^{n_n'} \sum_{n=1}^{n_n'} (k,n) z_{nm} I_m \qquad (k = 1,2,\ldots,n_{vs})$$

$$\det{}' = \det(\tilde{z}_{kl}) \text{ assumed } \neq 0 \quad (\text{with } k \,\&\, l = 1,2,\ldots,n_{vs})$$

$$y_{lk} = \operatorname{cof} \tilde{z}_{kl} / \det{}'. \qquad (l \,\&\, k = 1,2,\ldots,n_{vs}).$$

Then from the first of the eqs. (3) we have:

$$U_n = \sum_{m=1}^{n_n'} z_{nm}\Big(\tilde{I}_m - \sum_{k=1}^{n_{vs}} (k,m) I_{vs_k}\Big). \qquad (n = 1,2,\ldots,n_n') \qquad (4)$$

Hence, substituting these expressions in the second of eqs. (3), we get:

$$\sum_{n=1}^{n_n'} (k,n) U_n = \sum_{n=1}^{n_n'} \sum_{m=1}^{n_n'} (k,n) z_{nm}\Big(\tilde{I}_m - \sum_{l=1}^{n_{vs}} (l,m) I_{vs_l}\Big) = -E_k \quad (k=1,\ldots,n_{vs})$$

that is:

$$\sum_{l=1}^{n_{vs}} \tilde{z}_{kl} I_{vs_l} = (E_k + \tilde{E}_k). \qquad (k = 1,2,\ldots,n_{vs}).$$

Consequently:

$$I_{vs_k} = \sum_{l=1}^{n_{vs}} \tilde{y}_{kl}(E_l + \tilde{E}_l) \qquad (k = 1,2,\ldots,n_{vs}); \qquad (5)$$

and substituting these expressions in eqs. (4) we obtain:

$$U_n = \sum_{m=1}^{n_n'} z_{nm}\Big[\tilde{I}_m - \sum_{k=1}^{n_{vs}} \sum_{l=1}^{n_{vs}} (k,m) \tilde{y}_{kl}(E_l + \tilde{E}_l)\Big]. \quad (n=1,2,\ldots,n_n'). \quad (6)$$

It can be checked that the U_n given by eqs. (2) satisfy eqs. (1), and that the U_n and I_{vs_k} given by eqs. (6 & 5) satisfy eqs. (3), as an exercise.

Even though the above formulas for the solution of the complex canonical nodal equations of a network are of great theoretical importance, in a given particular case of a specific network it is better (when $n_{vs} > 0$) to first solve the second of the eqs. (3) for n_{vs} of the complex nodal potentials U_n in terms of the (known) complex electromotive forces E_k of the voltage sources and the other complex nodal potentials, and then to eliminate them by substitution in the first of the eqs. (3). In this way we obtain a system of n_n'(=number of independent nodes of the given network) equations in n_n' unknowns. By solving these n_n' equations for the n_n' unknowns left in them (in practice the best way is by systematic successive eliminations) and then reversing the previous steps, all the unknown complex nodal potentials U_n $(n=1,2,\ldots,n_n')$ and complex currents I_{vs_k} $(k=1,2,\ldots,n_{vs})$ through the voltage sources can be obtained. They should then be substituted into the complex canonical nodal equations to check the results.

Once all the complex nodal potentials U_n and all the complex currents I_{vs_k} have been obtained (and checked), all the complex voltage drops V_k $(k=1,2,\ldots,n_e)$ across the general parallel elements of the network may be obtained by means of eqs. (0) of the introduction to this chapter (or, in practice, by eqs. 1, §3); and then all the complex (terminal) currents I_k $(k=1,2,\ldots,n_e)$

entering (and leaving) the general parallel elements in the reference directions assigned to them may be obtained by eqs. (1) of §1.

Finally, all the complex currents and voltages can be transformed back to the actual (time dependent) sinusoidal currents and voltages of the same angular frequency as that of the sources, by means of the fundamental isomorphism between complex numbers and sinusoids of the same angular frequency, as explained in Ch. V, §2 (p. 150); and we may be sure that these sinusoidal currents and voltages shall satisfy the original (time dependent) integro-differential equations of the network as a consequence of the one-to-one correspondence between sinusoids of the same frequency and complex numbers which transforms the original integro-differential equations into the complex equations of the network and viceversa. However, this final step is usually omitted because it is considered trivial.

Problem. If $p=\sigma+i\omega$ is a generalized natural frequency (also called a natural complex frequency) of a network, the general equations of the network with all the exciting functions nullified shall be satisfied by response functions of the form $A\,e^{\sigma t}\sin(\omega t+\alpha)$, or of the form $A\exp(pt)$. Assuming all the network parameters to be constant and all the exciting functions to be null and all the currents and voltages to be exponentially modulated sinusoids of the above form, and then using the one-to-one correspondence between such functions and complex numbers considered in Ch. IV, §6 (in the manner used in Problem 4, Ch. IV, §6), show that if p is a generalized natural frequency of a (stationary) constant parameter network of general parallel elements we shall have (when $n_{vs}>0$ and assuming the elements with a voltage source to be numbered first):

$$\sum_{n=1}^{n'_n} y_{mn}U_n + \sum_{k=1}^{n_{vs}} (k,m) I_{vs_k} = 0 \quad (m=1,2,\ldots,n'_n) \qquad (7)$$

$$\sum_{n=1}^{n'_n} (k,n) U_n = 0 \qquad (k=1,2,\ldots,n_{vs}=n'_n)$$

where $y_{mn}=y_{mn}(p)$ are the nodal admittances at the generalized frequency p, namely: $y_{mn}=g_{mn}+c_{mn}p+\gamma_{mn}/p$ (see eqs. 6 of §2), and U_n $(n=1,2,\ldots,n'_n)$ and I_{vs_k} $(k=1,2,\ldots,n_{vs})$ are the complex numbers corresponding to the exponentially modulated nodal potentials and currents (through the nullified voltage sources), respectively. Finally, assuming that the above eqs. (7) are not identically satisfied, and that they have a non-vanishing solution, show that the generalized natural frequency, p, of the network must satisfy the following determinantal equation:

$$\begin{array}{c}
\begin{array}{c|cccccccc|}
\uparrow & y_{11} & y_{12} & \cdots & y_{1n'_n} & (1,1) & (2,1) & \cdots & (n_{vs},1) \\
n'_n \text{ rows} & y_{21} & y_{22} & \cdots & y_{2n'_n} & (1,2) & (2,2) & \cdots & (n_{vs},2) \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\downarrow & y_{n'_n1} & y_{n'_n2} & \cdots & y_{n'_nn'_n} & (1,n'_n) & (2,n'_n) & \cdots & (n_{vs},n'_n) \\
\uparrow & (1,1) & (1,2) & \cdots & (1,n'_n) & 0 & 0 & \cdots & 0 \\
n_{vs} \text{ rows} & (2,1) & (2,2) & \cdots & (2,n'_n) & 0 & 0 & \cdots & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\downarrow & (n_{vs},1) & (n_{vs},2) & \cdots & (n_{vs},n'_n) & 0 & 0 & \cdots & 0 \\
\multicolumn{1}{c}{} & \multicolumn{4}{c}{\leftarrow n'_n \text{ columns} \rightarrow} & \multicolumn{4}{c}{\leftarrow n_{vs} \text{ columns} \rightarrow}
\end{array}
\end{array}
= \left|\begin{array}{c|c}
\begin{array}{c} y_{mn}(p) \\ (m \& n=1,2,\cdots,n'_n) \end{array} & \begin{array}{c} (k,m)_{\text{Transposed}} \\ (k=1,2,\cdots,n_{vs}) \\ (m=1,2,\cdots,n'_n) \end{array} \\ \hline
\begin{array}{c} (k,n) \\ (k=1,2,\cdots,n_{vs}) \\ (n=1,2,\cdots,n'_n) \end{array} & 0
\end{array}\right| = 0. \qquad (8)$$

When the number n_{vs} of voltage sources in the network is zero, the determinantal equation satisfied by the generalized natural frequencies $\underline{p}$ *of the network reduces to the following equation:*

$$\det (y_{mn}) = |(y_{mn})| = |(g_{mn}+c_{mn}p + \gamma_{mn}/p)| = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n'_n} \\ y_{21} & y_{22} & \cdots & y_{2n'_n} \\ \vdots & \vdots & & \vdots \\ y_{n'_n1} & y_{n'_n2} & \cdots & y_{n'_nn'_n} \end{vmatrix} = 0. \qquad (9)$$

§5. EXISTENCE AND UNIQUENESS THEOREMS FOR THE NODE METHOD.

Consider an arbitrary network of n_e general parallel elements connected into n_c separate connected parts (components) with n_n nodes (i.e. with $n'_n=n_n-n_c$ independent nodes) and n_m independent meshes in which all the (current and voltage) sources have sinusoidal values of angular frequency $\omega \neq 0$. The general complex equations (1, 2, 3, §1) of the network shall have a solution if and only if the network has a sinusoidal solution of the same frequency as that of the sources; moreover, assuming the network to have only $n_{vs}(\leq n'_n)$ voltage sources, inserted in the elements 1, 2,..., n_{vs}, the complex canonical equations (2, §2) of the network shall have a solution if and only if the general complex equations have one (see the general comments given in Ch. VI, §5); hence the existence of sinusoidal solutions is equivalent to the existence of solutions of the complex canonical equations.

Also, the unicity of a sinusoidal solution is equivalent to the unicity of the solution of the general complex equations; because of the isomorphism between sinusoids of the same frequency and complex numbers which establishes a one to one correspondence between the sinusoidal currents and voltages and the complex currents and voltages (and the corresponding equations) of a network. Furthermore, the unicity of the solution of the general complex equations is equivalent to the unicity of the solution of the complex canonical equations. For it is clear that the *complex nodal potentials U_n $(n=1,2,\dots,n'_n)$ and the complex currents I_{vs_k} $(k=1,2,\dots,n_{vs})$ through the voltage sources (which are the unknowns in the complex canonical equations) determine the complex voltage drops V_k and the complex terminal currents I_k $(k=1,2,\dots,n_e)$ through the general parallel elements (which are the unknowns in the general complex equations) of the network uniquely, by eqs. (0) of the introduction to this chapter and eqs. (1) of §1. And the converse of this is also true. Because if there were two sets of U_n and I_{vs_k} (with differences ΔU_n and ΔI_{vs_k}) corresponding to a unique set of V_k and I_k, then from eqs. (1), §1, we would obtain by subtraction: $\Delta I_{vs_k}=0$, and from eqs. (0) of the introduction to this chapter we would obtain by subtraction:*

$$\sum_{n=1}^{n'_n}(k,n)\Delta U_n = 0. \qquad (k=1,2,\dots,n_e) \qquad (1)$$

Consequently:

$$\sum_{k=1}^{n_e}\Big(\sum_{n=1}^{n'_n}(k,n)\Delta U_n\Big) I_k = \sum_{n=1}^{n'_n}\Delta U_n\Big(\sum_{k=1}^{n_e}(k,n)I_k\Big) = 0. \qquad (2)$$

But the expressions $\sum_{k=1}^{n_e}(k,n)I_k$ are linearly independent (cf. Ch. II, §3), and so all their coefficients ΔU_n must vanish (cf. 2º, p. 56); hence the U_n and I_{vs_k} are uniquely determined by the V_k and I_k, too.

Now the complex canonical equations (2, §2 or 3, §4) of a given network of general parallel elements shall have a solution for arbitrary sinusoidal sources of the same angular frequency ω if and only if the determinant of the coefficients of the unknowns does not vanish, i.e.,

$$\Delta = \begin{vmatrix} y_{11}(i\omega) & y_{12}(i\omega) & \dots & y_{1n'_n}(i\omega) & (1,1) & (2,1) & \dots & (n_{vs},1) \\ y_{21}(i\omega) & y_{22}(i\omega) & \dots & y_{2n'_n}(i\omega) & (1,2) & (2,2) & \dots & (n_{vs},2) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_{n'_n1}(i\omega) & y_{n'_n2}(i\omega) & \dots & y_{n'_nn'_n}(i\omega) & (1,n') & (2,n') & \dots & (n_{vs},n') \\ (1,1) & (1,2) & \dots & (1,n'_n) & 0 & 0 & \dots & 0 \\ (2,1) & (2,2) & \dots & (2,n'_n) & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (n_{vs},1) & (n_{vs},2) & \dots & (n_{vs},n'_n) & 0 & 0 & \dots & 0 \end{vmatrix} \neq 0, \qquad (3)$$

(where the $y_{mn}(i\omega)$ are given by eqs. (1), §2)

the last n_{vs} rows and n_{vs} columns being omitted when $n_{vs}=0$; and then (and only then) the solution shall also be unique for given sources. Networks for which relation (3) holds are called <u>non-singular networks</u> (of general parallel elements), and when no mention is made to the contrary we will always assume the networks considered to be non-singular (which is no practical limitation; see remarks on p. 179 and the note in Ch. XII, §7).

(Note that ω can not be a natural angular frequency of the network when (3) holds.)

When the relation (3) holds for a given network, the unique values of the unknown complex nodal potentials U_n $(n=1,2,\ldots,n'_n)$ and complex currents I_{vs_k} $(k=1,2,\ldots,n_{vs})$ through the voltage sources can be obtained by Cramer's rule, namely:

$$U_n = \Delta_n/\Delta \quad (n=1,2,\ldots,n'_n), \qquad I_{vs_k} = \Delta_{n'_n+k}/\Delta \quad (k=1,2,\ldots,n_{vs}), \quad (4)$$

where Δ_l $(l=1,2,\ldots,n'_n,n'_n+1,\ldots,n'_n+n_{vs})$ is the determinant obtained from the determinant Δ given above in (3) by replacing the elements of the $l^{\underline{th}}$ column by the elements $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_{n'_n}, -E_1, -E_2, \ldots, -E_{n_{vs}}$ (i.e. by replacing the $l^{\underline{th}}$ column in Δ by the column of the independent terms in the eqs. 3 of §4). The $\tilde{I}_n$ $(n=1,2,\ldots,n'_n)$ are given by the first of the eqs. (3) of §4, and the E_k $(k=1,2,\ldots,n_{vs})$ are the complex electromotive forces of the voltage sources.

Note that in non-singular networks, the only possible solution when all the sources have null values is the trivial null solution (with all the currents and voltages = 0). Because all the Δ_l in eqs. (4) are then zero, since each will then have a column of zeros.

When the relation (3) does not hold for a given network, any solution that may exist shall not be unique at any rate (see the remarks on p. 180); and a solution shall exist if and only if the rank of the square array within the determinant Δ given in (3) is the same as the rank of the augmented table obtained from it by annexing a column with the entries: $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_{n'_n}, E_1, E_2, \ldots, E_n$ (which is equivalent to say that the complex canonical equations of the network are linearly dependent for the particular sources given). If $\underline{r}(<n'_n+n_{vs})$ be the common rank, then (some) $\underline{r}$ of the complex canonical equations of the network are linearly independent (and no more) while all the rest are linearly dependent on them; and the values of (some set of) $n'_n+n_{vs}-r$ unknowns are arbitrary while the rest can be determined linearly in terms of them by Cramer's rule, from the $\underline{r}$ linearly independent equations; moreover, the unknowns thus found will satisfy all the complex canonical equations of the network.

Of course, in any specific case, all this (whether the equations have a solution or not, the rank of the system, which equations are linearly independent and which depend on them, which unknowns are arbitrary, and the solution itself, if any exists) can best be investigated by Crout's process, as explained at the end of the appendix to Chapter VI.

<u>Problem</u>. Show that: $\det=\det(y_{mn})\neq 0$ & $\det'=\det(z_{kl})\neq 0$ (see §4) implies $\Delta\neq 0$, and if $n_{vs}<n'_n$, $\Delta\neq 0$ implies $\det\neq 0$ & $\det'\neq 0$. [Hint: Cf. end Ch. VI, §5]

§6. THE SUPERPOSITION PRINCIPLE.

The superposition principle for an arbitrary (stationary constant parameter) network in the sinusoidal state can also be established by taking it (as can always be done) as a network of general parallel elements, with the general complex canonical eqs. (3), §4, whose solution is given by eqs. (4), §5, when the network is non-singular. This undertaking is unnecessary of course, because the network may also be taken as a network of general series elements, for which the superposition principle has already been established in Ch. VI. However, it may be instructive to do so, even if only briefly.

In the first place we develop each of the determinants Δ_l $(l=1,2,\ldots,n_n'+n_{vs})$ *in eqs. (4), §5, according to the corresponding* l^{th} *column. In this way we obtain each of the complex nodal potentials* U_n $(n=1,2,\ldots,n_n')$ *and each of the complex currents through the voltage sources,* I_{vs_k} $(k=1,2,\ldots,n_{vs})$*, as homogeneous linear functions of the* $\tilde{I}_n$ $(n=1,2,\ldots,n_n')$ *and the* E_k $(k=1,2,\ldots,n_{vs})$*. Then replacing the* $\tilde{I}_n$ *by their expressions given in the first of eqs. (3), §4, we can obtain all the* U_n *and* I_{vs_k} *as homogeneous linear functions of the complex currents* I_{cs_k} $(k=1,2,\ldots,n_e)$ *through the current sources and the complex electromotive forces* E_k $(k=1,2,\ldots,n_{vs})$ *of the voltage sources of the network. In short, each* U_n *and* I_{vs_k} *is obtained as a linear combination of the complex source values of the network.*

By eqs. (0) of the introduction to this chapter, and eqs. (1) of §1, we can then obtain each of the complex voltage drops V_k *and complex terminal currents* I_k *in the elements* $\underline{k}(=1,2,\ldots,n_e)$ *of the network as a linear combination of the complex values of its sources.*

Then, from these results (as was done in Ch. VI, §6), we can infer that each U_n, I_{vs_k}, V_k, I_k *(i.e., each complex current and voltage) of the given network can be obtained by superposing (i.e., adding) the corresponding quantities in the auxiliary networks obtained from the given network by nullifying all the sources except one at a time. This is the* ***first form of the superposition principle.***

It can also be inferred that each complex current and voltage of the given network can be obtained by adding the corresponding quantities of the auxiliary networks obtained from the given network first by arbitrarily grouping all the sources into mutually exclusive classes and then by nullifying all the sources except those of one of these mutually exclusive classes at a time. This is the ***second form of the superposition principle****, which clearly includes the first.*

CHAPTER VIII: NETWORKS OF TWO-TERMINAL STRUCTURES.

In the preceding chapters we have given the general theory of networks of general series elements and of general parallel elements. As we have said before, any network may be considered either as a network of general series elements or as a network of general parallel elements, so that any further considerations are not strictly necessary. However, it is sometimes convenient to be able to consider more general consistent inter-connected collections of basic elements with two available mutually accessible terminals (by which ~~it~~ they may be connected to other such things) as the generic elements of a network. Such consistent inter-connected collections (networks) of basic elements with two available terminals (considered as units) will be called two-terminal structures or boxes. When no source (with non-zero value) is included in a box, the structure will be called passive, otherwise it will be called active. Of course, a given collection of basic elements inter-connected in different ways, or with distinct pairs of mutually accessible nodes chosen as terminals (even if they are inter-connected in the same way) gives rise to distinct boxes.

In this chapter we propose to give the general theory of (stationary constant parameter) networks of such two-terminal structures in the sinusoidal state. Our previous results will then be particular cases of the results of this chapter.

§1. THE GENERAL CONCEPTS OF IMPEDANCE AND ADMITTANCE.

Consider an arbitrary passive two-terminal box with the mutually accessible terminals a and b (see Fig. 0). Let us assume that a

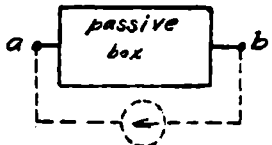


Fig. 0.

sinusoidal voltage source of angular frequency ω is connected between the terminals a and b, as indicated by the dotted lines in the figure. Since the terminals a and b are (assumed to be) mutually accessible through (elements within) the box, we can be sure that the voltage source will be an essential element in the augmented network formed by the box and the source; i.e., the source will belong to a certain mesh in the augmented network, which we will take as the mesh 1, and in which it may be taken as the key element. Now let us choose within the box a complete independent set of meshes, which we will number $2, 3, \ldots, n_m$; then the set of meshes $1, 2, \ldots, n_m$ shall be a complete independent set of meshes in the augmented network (cf. Ch. II, §4). Since this network has a single source, namely a voltage source, of complex value $\underline{E}$ (say), which belongs only to the mesh 1, the complex canonical mesh equations of the network may be written as follows:

$$\sum_{n=1}^{n_m} z_{1n} J_n = E,$$

$$\sum_{n=1}^{n_m} z_{mn} J_n = 0, \quad (m = 2, 3, \ldots, n_m) \tag{1}$$

where the z_{mn} are the mesh impedances, and the J_n are the circulating currents of the meshes $1, 2, \ldots, n_m$, and we know that J_1 is the complex current entering the box through the voltage source.

If $\det(z_{mn})$ does not vanish, the above equations (1) may be solved uniquely for the complex mesh currents J_n $(n = 1, 2, \ldots, n_m)$. The only one we are interested in now is J_1, for which we have:

$$J_1 = E \operatorname{cof} z_{11} / \det(z_{mn}). \tag{2}$$

If our box is part of an arbitrary network in the sinusoidal state (at the angular frequency ω), let $\underline{V}$ be the complex voltage drop, and $\underline{I}$ the complex current, through the box (in the same arbitrarily chosen reference direction of the mesh 1), and let J'_n $(n = 2, 3, \ldots, n_m)$ be the complex mesh currents in the meshes $2, 3, \ldots, n_m$ of the box. The complex currents $\underline{I}$ and J'_n $(n = 2, 3, \ldots, n_m)$ substituted for the J_n $(n = 1, 2, \ldots, n_m)$ will satisfy the same eqs. (1), for $m = 2, 3, \ldots, n_m$ with the same coefficients z_{mn}; and the complex voltage drop $\underline{V}$ will be given precisely by the first member of the first of eqs. (1), also with the same z_{mn} (note that z_{11} is the self impedance of the mesh 1 with the terminal $\underline{a}$ and $\underline{b}$ of the box short circuited); and so under the conditions assumed above, the complex current $\underline{I}$ will be given in terms of the complex voltage drop $\underline{V}$ uniquely by eq. (2), with $\underline{E}$ and J_1 replaced by $\underline{V}$ and $\underline{I}$, respectively.

Hence, in any case, for a given angular frequency, the ratio of the complex voltage drop across the box to the complex current through it will be constant, namely:

$$V/I = E/J_1 = \det(z_{mn}) / \operatorname{cof} z_{11}. \tag{3}$$

This constant of the box is called its impedance at the angular frequency ω; and the inverse is called its admittance. It can be checked that all our previous uses of the words (self) impedance and admittance are consistent with the present use of the words. (See Prob. 1, below.)

Denoting the impedance of a box by $\underline{Z}$, and its admittance by $\underline{Y}$, the complex voltage drop $\underline{V}$, and the complex current $\underline{I}$, through it (in the same reference direction) shall then be related by the equations:

$$V = ZI \qquad \text{and} \qquad I = YV, \tag{4}$$

where (in the notation given above):

$$Z = \det(z_{mn})/\operatorname{cof} z_{11} \qquad \text{and} \qquad Y = 1/Z = \operatorname{cof} z_{11}/\det(z_{mn}). \tag{5}$$

Except for the explicit expressions (5) for the impedance and admittance of the given box, the above results can also be obtained by using the existence and uniqueness theorems, and the superposition principle, by what may be called a *guess and check method*. If the box is connected in a network, and V is the complex voltage drop across it (from the termianal a to the terminal b, say), let it first be removed from the network and then connected to a voltage source of complex voltage rise E (from b to a) equal to V. In this situation, under the conditions assumed above, the existence and uniqueness of the complex currents and voltages in the box is guaranteed; and so they shall be the same as when the box was connected in the network. (We guessed that they were the same, and that they are is proved by this check.) Finally, the proportionally between the complex current entering the box and the complex voltage drop $V=E$ across it follows from the superposition principle (for a single source).

Both the impedance Z and the admittance Y of a given box depend on the angular frequency ω. The real part of Z is called the *equivalent* or *effective resistance* (or *resistive part*) of the box (and of its impedance) at the given frequency, and its imaginary part is called the *equivalent* or *effective reactance* (or *reactive part*) of the box (and of its impedance), at the given frequency. When the equivalent reactance of a box is positive at a given frequency, the box and its impedance are said to be *inductive at the given frequency*; and when it is negative, they are said to be *capacitive at the given frequency*. When the equivalent reactance is zero, the box and its impedance are said to be (purely) *resistive at the given frequency*.

Similarly, the real part of the admittance Y is called the *equivalent* or *effective conductance* (or *conductive part*) of the box (and of its admittance) at the given frequency, and its imaginary part is called the *equivalent* or *effective susceptance* (or *susceptive part*) of the box (and of its admittance), at the given frequency. When the equivalent susceptance of a box is positive at a given frequency, the box and its admittance are said to be *capacitive at the given frequency*; and when it is negative, they are said to be *inductive at the given frequency*. Otherwise, they are said to be (purely) conductive at the given frequency.

A generic name for an impedance and an admittance of a two-terminal structure or box is the useful word: immittance (or the word: adpedance; see, e.g., Bode, Network Analysis and Feedback Amplifier Design, van Nostrand, 1945, p. 15). These terms are sometimes used as synonyms of the word box as used here.

Problem 1. Compute the impedances and admittances of the two-terminal structures shown in Fig. 1.

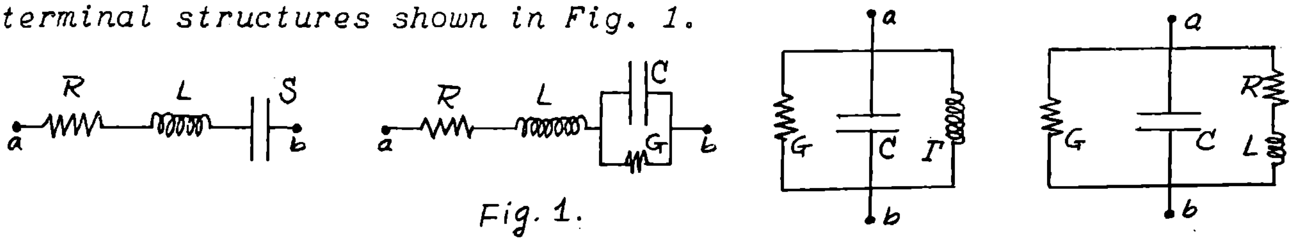


Fig. 1.

Problem 2. Let R be the equivalent resistance, and X the equivalent reactance of a box with the impedance Z. Let Y be its admittance, and G and B its equivalent conductance and susceptance, respectively (all considered at the same angular frequency). Show that:

$$\begin{aligned} Y &= 1/Z = \bar{Z}/|Z|^2 = \frac{R}{R^2+X^2} - \frac{iX}{R^2+X^2} = G + iB, \\ G &= R/(R^2+X^2), \qquad B = -X/(R^2+X^2), \\ Z &= 1/Y = \bar{Y}/|Y|^2 = \frac{G}{G^2+B^2} - \frac{iB}{G^2+B^2} = R + iX, \\ R &= G/(G^2+B^2), \qquad X = -B/(G^2+B^2). \end{aligned} \tag{6}$$

From these formulae, show that an impedance is inductive, capacitive, or resistive, if and only if the admittance is inductive, capacitive, or conductive, respectively. Thus the kind of an immittance is uniquely defined.

Problem 3. If Z_k $(k = 1,2,\dots,n)$ are the impedances of n boxes connected in series as shown in Fig. 2, show that the impedance Z of the combination is the sum of the impedances: $Z = Z_1+Z_2+\dots+Z_n$. (7)
The admittance of the series combination is then: $Y = Z^{-1} = \dfrac{Y_1Y_2\cdots Y_n}{Y_1\cdots Y_{n-1}+\dots+Y_2\cdots Y_n}$.

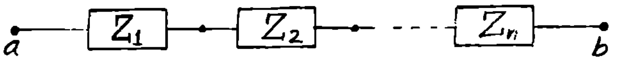


Fig. 2.

Problem 4. If Y_k $(k = 1,2,\dots,n)$ are the admittances of n boxes connected in parallel as shown in Fig. 3, show that the admittance Y of the combination is the sum of the admittances: $Y = Y_1+Y_2+\dots+Y_n$. (8)
The impedance of the parallel combination is then: $Z = Y^{-1} = \dfrac{Z_1Z_2\cdots Z_n}{Z_1\cdots Z_{n-1}+\dots+Z_2\cdots Z_n}$.

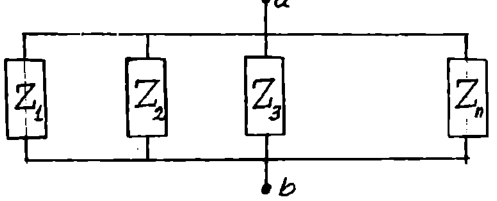
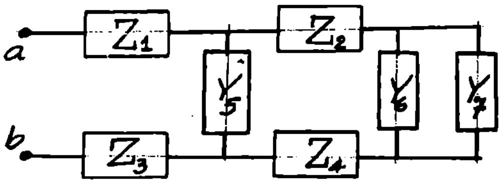
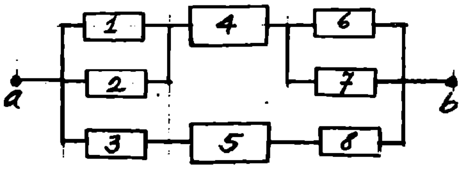


Fig. 3.

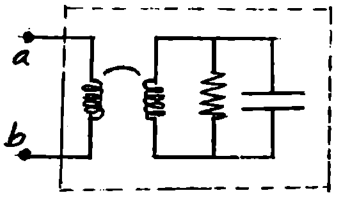
Problem 5. Using the results of Prob. 3 & 4, find the impedances of the following structures with the terminals a and b. (Figs.4.)



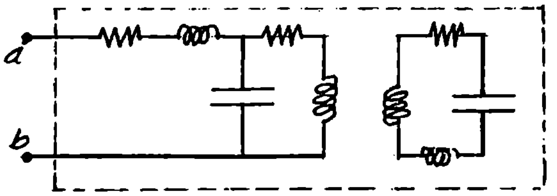
Figs. 4.



Problem 6. Compute the impedances of the following structures with the terminals a and b. (Figs.5) (Hint: Establish the complex canonical equations of the network obtained by connecting a voltage or current source between the terminals a and b, and then find the ~~quotient~~ ratio of the complex voltage rise to the complex current through the source; this is the impedance.)



Figs. 5.



In some cases, when a given two-terminal structure contains many elements of the general parallel type, it is better to compute the impedance and the admittance of the box (at a given frequency) by making use of the complex canonical nodal equations instead of the mesh equations (1). If the box has n_n nodes in n_c separate parts (=components), it shall have $n_n' = n_n - n_c$ independent nodes only. Assume that two mutually accessible nodes a and b (on the same component) of the box are taken as the two terminals of the box, and let us connect a sinusoidal (voltage or current) source of the given frequency between them. Taking the node a as the node 1 and the node b as the base node in the corresponding component, and denoting the complex current, and the complex voltage rise, through the source (from b to a) by I and E, respectively, the complex canonical nodal equations of the augmented network formed by the box and the source will be the following:

$$\sum_{n=1}^{n_n'} y_{1n}U_n = I, \qquad \sum_{n=1}^{n_n'} y_{mn}U_n = 0 \quad (m = 2, 3, \ldots, n_n'), \qquad U_1 = E; \qquad (9)$$

where the y_{mn} are the nodal admittances and the U_n are the complex nodal potentials (referred to the corresponding base nodes).

If $\det(y_{mn})$ does not vanish, the eqs. (9) may be solved uniquely for the complex nodal potentials U_n as follows:

$$U_n = I \operatorname{cof} y_{1n} / \det(y_{mn}). \qquad (10)$$

The only one of these we are interested in (to find the impedance Z and the admittance $\underline{Y}$ of the box) is U_1; and we have:

$$Z = 1/Y = E/I = U_1/I = \operatorname{cof} y_{11}/\det(y_{mn}). \qquad (11)$$

In any specific case, the impedance and the admittance of a given passive box whose internal structure is known can best be computed by obtaining this expression (11), or the expression (3), from the complex canonical equations by Crout s method.

Of course, the impedance and admittance of a two-terminal passive box at a given frequency can also be obtained experimentally (without a knowledge of the internal structure) by connecting a sinusoidal source of the same given frquency between its two terminals, and then measuring the effective voltage rise $|E|$ and the effective current $|I|$ through the source and the phase angle, $\vartheta =$ ang E - ang I , between them. The impedance $\underline{Z}$ and the admittance $\underline{Y}$ are then:

$$Z = \frac{|E|}{|I|}\angle\vartheta \qquad \text{and} \qquad Y = \frac{|I|}{|E|}\angle{-\vartheta}. \qquad (12).$$

<u>Definition</u>. When a passive two-terminal box is taken as a network of elements of the general series type and the determinant of the impedances of a complete independent set of meshes in the box with its two terminals connected together does not vanish, or when the two-terminal box is taken as a network of elements of the general parallel type and the determinant of the nodal admittances does not vanish, then (and only then) the box will be called a <u>non-singular box</u>.

§2. SYSTEMS OF TWO-TERMINAL PASSIVE STRUCTURES.

Let us now consider a set of $\underline{N}$ magnetically coupled two-terpassive structures, numbered 1,2,...,N, as shown in Fig. 1. We again assume that the terminals of each box are mutually accessible through elements within the corresponding box, and that a reference direction is arbitrarily assigned to each box from one of its terminals to the other. Suppose that sinusoidal (voltage or current) sources of the same angular frequency ω are connected across the terminals of each box (as shown by the dotted lines in Fig. 1), and let V_k and I_k denote the complex voltage drop and cur-

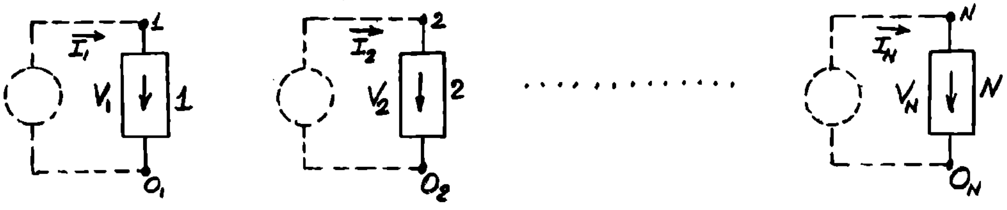


Fig. 1.

rent (respectively) from one terminal to the other through the box $\underline{k}$ *in the assigned reference direction. In the resulting network formed by the boxes and the sources connected across their terminals, each of the sources will be an essential element. Taking the source connected across the box* $\underline{k}$ *as a key element, let us complete a mesh through elements of the box* $\underline{k}$, *which will be taken as the mesh* $\underline{k}$ $(=1,2,\ldots,N)$ *oriented in the same sense as the box. Besides, let us choose a complete and independent set of meshes within the* $\underline{N}$ *boxes, which will be numbered* $N+1, N+2,\ldots,n_m$. *Then the set of meshes* $1,2,\ldots,n_m$ *will be a complete and independent set of meshes in the network formed by the boxes and sources, the complex canonical mesh equations of which are:*

$$\sum_{n=1}^{n_m} z_{mn}J_n = V_m \qquad (m=1,2,\ldots,N) \qquad \text{(1a)}$$

$$\sum_{n=1}^{n_m} z_{mn}J_n = 0 \qquad (m=N+1,\ldots,n_m) \qquad \text{(1b)}$$

$$J_k = I_k \qquad (k=1,2,\ldots,N) \qquad \text{(1c)}$$

where the J_n *are the complex circulating currents around the meshes* $n\,(=1,2,\ldots,n_m)$ *and the* z_{mn} *are the mesh impedances. (In particular, notice that for* $m \leqq N$, z_{mm} *is the impedance of the mesh* $\underline{m}$ *with the terminals of the box (m) connected together.)*

When the determinant of the mesh impedances does not vanish, the system of eqs.(1a,b) shall be consistent and linearly independent, and then the complex mesh currents J_n $(n=1,2,\ldots,n_m)$, *and therefore all the complex currents in the elements of the boxes, shall be determined uniquely by the complex voltage drops* V_k $(k=1,2,\ldots,N)$ *across the boxes, and viceversa. If, furthermore, the minor in the determinant of the mesh impedances corresponding to the last* n_m-N *rows (or equations) and columns (or unknowns) also does not vanish, then the system of eqs. (1b,c) shall also be consistent and linearly independent, and then the complex mesh currents* J_n $(n=1,2,\ldots,n_m)$ *will also be determined uniquely by the terminal complex currents* I_k $(k=1,2,\ldots,N)$ *traversing the boxes, and viceversa. Consequently, when both these conditions are satisfied, the complex voltage drops* V_k $(k=1,2,\ldots,N)$ *across the boxes shall be determined uniquely by the complex terminal currents* I_k $(k=1,2,\ldots,N)$ *through the boxes, and viceversa. (In any case, when the second of these conditions is satisfied, it can be seen that at least the complex voltage drops* V_k *across the boxes will be determined uniquely by the complex terminal currents* I_k *through them.)*

When the determinants $\det(z_{mn})$ for m & $n = 1,2,\ldots,n_m$, and $\det(z_{mn})$ for m & $n = N+1, N+2,\ldots,n_m$, do not vanish, the system of $\underline{N}$ boxes will be called a <u>non-singular system of boxes</u> (at the angular frequency ω); and, unless otherwise explicitly stated, we will always assume this to be the case. What we have proved above may now be stated as follows. In a non-singular system of boxes, the complex voltage drops across them are determined uniquely by the complex terminal currents traversing them, and viceversa.

The explicit expressions for the complex voltage drops V_k across the boxes in terms of the complex terminal currents I_k traversing them $(k = 1,2,\ldots,N)$ can be found from eqs. (1) by eliminating the complex mesh currents J_n, in which we are no longer (theoretically) interested. Putting $\det = \det(z_{mn})$ for m & $n = N+1,\ldots,n_m$, and:

$$\tilde{y}_{mn} = \operatorname{cof} z_{nm} / \det \qquad (m \,\&\, n = N+1,\ldots,n_m), \qquad (2)$$

we obtain from equations (1b) and (1c):

$$J_n = -\sum_{m=1+N}^{n_m} \sum_{l=1}^{N} \tilde{y}_{nm} z_{ml} I_l \qquad (n = N+1,\ldots,n_m). \qquad (3)$$

Hence, substituting in equations (1a), we obtain:

$$V_k = \sum_{l=1}^{N} Z_{kl} I_l \qquad (k = 1,2,\ldots,N), \qquad (4)$$

where:

$$Z_{kl} = z_{kl} - \sum_{m=N+1}^{n_m} \sum_{n=N+1}^{n_m} z_{km} \tilde{y}_{mn} z_{nl} \qquad (k \,\&\, l = 1,2,\ldots,N). \quad (5)$$

In any practical case, it is best to obtain eqs. (3) directly from eqs. (1b) by Crout's process, after replacing the J_n for $n \leqq N$ by I_n.

Eqs. (4) may be solved for the complex terminal currents I_k (also) as homogeneous linear functions of the complex voltage drops V_k across the boxes, thus:

$$I_k = \sum_{l=1}^{N} Y_{kl} V_l \qquad (k = 1,2,\ldots,N), \qquad (6)$$

where:

$$Y_{kl} = \operatorname{cof} Z_{lk} / \det(Z_{kl}) \qquad (k \,\&\, l = 1,2,\ldots,N). \qquad (7)$$

And, of course, we also have:

$$Z_{kl} = \operatorname{cof} Y_{lk} / \det(Y_{kl}) \qquad (k \,\&\, l = 1,2,\ldots,N). \qquad (8)$$

In any practical case, these quotients of the cofactors divided by the corresponding determinants can best be obtained directly by Crout's process from equations (4) and (6), respectively.

Now assume that our non-singular system of $\underline{N}$ boxes is part of a network in the sinusoidal state at the angular frequency ω, and let the complex currents traversing them be I_k $(k = 1,2,\ldots,N)$. We assume that our system includes all boxes that are magnetically coupled (i.e., that the system is magnetically closed). We can be sure that the complex voltage drops across the boxes will still be given by eqs. (4). Because, if these were V'_k, we could disconnect the boxes from the network and connect sources producing complex voltage drops V'_k and complex currents I_k through them, as can easily be proved by the guess and check method (§1, p. 217); but since the complex terminal currents I_k determine the complex voltage drops through them uniquely, we must have $V'_k = V_k$ $(k=1,\ldots,N)$. Hence, whether the non-singular system of boxes is part of a network or not, the complex voltage drops and currents through them are related by the uniquely determined homogeneous linear relationships (4) and (6), once the reference directions are assigned.

The categorically determined coefficients, Z_{kl} $(k\,\&\,l = 1,\ldots,N)$, in the homogeneous linear relations (4), which are in general complex functions of the angular frequency ω of the applied sources (and in fact, explicit functions of ωi), are called the (self- and mutual-) impedances of the system of boxes. Z_{kk}, which is often written simply Z_k in practice, is called the self-impedance of the box k in the presence of all the other boxes of the system (or simply its impedance), and should carefully be distinguished from the impedance of the box by itself, i.e. isolated, not an element of a system of boxes. Likewise, Z_{kl} is called the mutual-impedance of (or between) the boxes k and l (at the angular frequency ω) in the presence of all the other boxes of the system, and it should carefully be distinguished from their mutual-impedance when they form a system of two boxes alone. (Z_{kk} may be considered as the mutual-impedance of the box k with itself, in the presence of the other boxes of the system.) Eqs. (5) give us the self- and mutual-impedances of a system of boxes in terms of their internal structural quantities.

Similarly, the categorically determined coefficients, Y_{kl} $(k\,\&\,l = 1,2,\ldots,N)$, in the homogeneous linear relations (6), are called the (self- and mutual-) admittances of the system of boxes, at the given angular frequency ω. Y_{kk}, often written simply Y_k in practice, is called the self-admittance (or simply admittance) of the box k, or its mutual-admittance with itself, in the presence of all the other boxes of the system; and it should carefully be distinguished from the admittance of the box by itself, i.e. when it is not part of a system of boxes; in particular then, in general

Y_k is not the inverse of Z_k, as it is when the box k is magnetically isolated. Likewise, Y_{kl} is called the mutual-admittance of (or between) the boxes k & l (at the angular frequency ω) in the presence of all the other boxes of the system, and it should carefully be distinguished from their mutual-admittance when they are alone. Of course, in general Y_{kl} is not the inverse of Z_{kl}, not even when the two boxes k & l form a system by themselves (as can be seen from eqs. (7), which give the Y_{kl} in terms of the Z_{kl}). (See Prob. 3 below.)

We again caution the beginner to distinguish between a box (or a pair of boxes) when it is part of a system and when it is not.

Problem 1. Show that: $\sum_{l=1}^{N} Y_{kl}Z_{lh} = \sum_{l=1}^{N} Z_{kl}Y_{lh} = \delta_{kl} = \begin{cases} 1 \text{ if } k=h \\ 0 \text{ if } k\neq h \end{cases}$.

Problem 2. From eqs. (5) show that $Z_{kl} = Z_{lk}$ for all k & l; and then from eqs. (7) show that $Y_{kl} = Y_{lk}$ for all k & l. This is, in essence, one form of the reciprocity theorm for networks of two-terminal passive structures (boxes). See **Ch. IX**, §1.

Problem 3. Show that in a system of two boxes ($N=2$), we have:
$Y_1 = Z_2/\det$, $Y_2 = Z_1/\det$, $Y_{12} = Z_{21}/\det = Y_{21}$, where we have put: $\det = Z_1Z_2 - Z_{12}Z_{21} = Z_1Z_2 - Z_{12}^2$.

Problem 4. Show that when for a certain k, $Z_{kl} = 0$ for all $l \neq k$ and $Z_k \neq 0$, we shall have: $\det(Z_{kl}) = Z_k \operatorname{cof} Z_{kk}$. Thence infer that for such a k, we shall have: $Y_k = 1/Z_k$. Consequently, the self-impedance and the self-admittance of a box not magnetically coupled with the other boxes of a system are reciprocal to each other and coincide, respectively, with the impedance and admittance of the box when it is by itself, as defined earlier in §1.

Problem 5. Considering that each mesh impedance z_{mn} in eqs. (2 & 5) is of the form: $R+i\omega L+S/i\omega$, show that each box immittance (impedance Z_{kl} and admittance Y_{kl}) is a rational function of ωi.

All the terminology established for immittances of isolated two-terminal structures (boxes) in §1, is also used for boxes of a system. Thus we speak of the equivalent resistance (or resistive part), the equivalent conductance (or conductive part), the equivalent reactance (or reactive part), and of the equivalent susceptance (or susceptive part) of a (self- or mutual-) immittance (=impedance or admittance), at a given frequency; and we also speak of an immittance of being capacitive, inductive, or purely resistive (or conductive) at a given frequency; just as we did in §1 for isolated boxes.

In some cases it is better to compute the impedances and the admittances of a system of boxes (at a given frequency) by using the complex canonical nodal equations instead of the mesh equations (1). Considering the network shown in Fig. 1 again, let n_n *be the total number of nodes, and* n_c *the total number of components, in the* $\underline{N}$ *boxes. Then the number of independent nodes shall be* $N'=n_n-n_c$. *Let us choose one terminal of the box* $\underline{k}$ $(=1,2,\ldots,N)$ *as a base node* O_k *and the other terminal as the node* $\underline{k}$*, as shown in Fig. 1; and, after omitting exactly any one node in every other component, let us (arbitrarily) number the rest of the nodes consecutively from* $N+1$ *to* N'.

The complex canonical nodal equations of the network are then:

$$\sum_{n=1}^{N'} y_{mn}U_n = I_m \qquad (m=1,2,\ldots,N) \tag{9a}$$

$$\sum_{n=1}^{N'} y_{mn}U_n = 0 \qquad (m=N+1,\ N+2,\ldots,N') \tag{9b}$$

$$U_n = V_n \qquad (n=1,2,\ldots,N) \tag{9c}$$

where the U_n *are the complex nodal potentials (voltage drops from the nodes* $\underline{n}$ *to the corresponding base nodes in the same components) and the* y_{mn} *are the nodal admittances. (Notice that for* $n \leqq N$, y_{nn} *is the self-admittance of the terminal* $\underline{n}$ *of the box* $\underline{n}$ *with its terminals loose.)*

Substituting the complex potentials U_n $(n \leqq N)$ *of the first* $\underline{N}$ *nodes, given by eqs. (9c), into eqs. (9b) and solving for the other complex potentials, we get:*

$$U_n = -\sum_{m=N+1}^{N'}\sum_{l=1}^{N} \tilde{z}_{nm}y_{ml}V_l \qquad (n=N+1,\ldots,N') \tag{10}$$

where we have put, for m & $n = N+1,\ N+2,\ldots,N'$:

$$\det = \det(y_{mn}) \text{ with } m\,\&\,n=N+1,\ldots,N',\ \& \quad \tilde{z}_{mn} = \operatorname{cof} y_{nm}/\det. \tag{11}$$

Substituting the expressions (10) into eqs. (9a) we then get:

$$\sum_{l=1}^{N}\Big(y_{kl} - \sum_{m=N+1}^{N'}\sum_{n=N+1}^{N'} y_{km}\tilde{z}_{mn}y_{nl}\Big)V_l = I_k \qquad (k=1,2,\ldots,N). \tag{12}$$

Comparing these equations with eqs. (6) we obtain for the admittances of the boxes of the system (self- and mutual-):

$$Y_{kl} = y_{kl} - \sum_{m=N+1}^{N'}\sum_{n=N+1}^{N'} y_{km}\tilde{z}_{mn}y_{nl} \qquad (k\,\&\,l=1,2,\ldots,N). \tag{13}$$

The (self- and mutual-) box impedances can then be found by eqs. (8)

We might add that, once the complex voltage drops and currents through the boxes of a system have been obtained, the internal complex currents and voltages may be obtained either by first finding the complex mesh currents from eqs. (1a,b) (with which the internal *branch complex currents, and then with these the complex voltage drops, may be found in the usual way) or else by first finding the complex nodal potentials from eqs. (9a,b) (with which the internal complex voltage drops, and then with these the complex currents, may be found in the usual way).*

The self- and mutual- immittances of a system of boxes (at a given angular frequency ω) can also be determined (measured) experimentally, without a detailed knowledge of the internal structures of the boxes. The basis of the technique to measure (self- and mutual-) impedances of boxes of a system is the system of equations (4); and the basis of the technique to measure (self- and mutual-) admittances is the system of eqs. (6). From eqs. (4) we see that if we open-circuit all boxes (i.e. disconnet their terminals) except (those of) box $\underline{l}$ (=1,2, ...,N) to which we connect a (sinusoidal current or voltage) source of angular frequency ω, and if we measure the complex current I_l through it, and the complex voltage drops it produces across the terminals of each of the boxes $\underline{k}$ (= 1,2,...,N), V_{kl} (say), in the assigned reference directions, then we will have:

$$Z_{kl} = V_{kl}/I_l \tag{14}$$

for each $\underline{k}$ and the given $\underline{l}$. Of course, if we repeat the above operation for each $\underline{l}$ we can determine all the Z_{kl} (k & l = 1,2,...,N). Naturally, it is only necessary to measure the effective values of V_{kl} and I_l and the phase angle between them, θ_{kl} (say) = ang V_{kl} − ang I_l, since:

$$Z_{kl} = V_{kl}/I_l = \left(|V_{kl}|/|I_l|\right)\underline{/\theta_{kl}}. \tag{15}$$

Similarly, from eqs. (6) we see that if we short-circuit all boxes (i.e. connect their terminals together) except (those of) the box $\underline{l}$ (=1,2,...,N) to which we connect a (sinusoidal current or voltage) source of angular frequency ω, and if we measure the complex voltage drop V_l across its terminals, and the induced complex currents I_{kl} (say) through each of the boxes $\underline{k}$ (=1,2,...,N) in the reference directions arbitrarily assigned to them, then we will have:

$$Y_{kl} = I_{kl}/V_l = \left(|I_{kl}|/|V_l|\right)\underline{/\theta'_{kl}} \qquad (k \,\&\, l = 1,2,\ldots,N), \tag{16}$$

where θ'_{kl} = ang I_{kl} − ang V_l is the phase angle between V_l and I_{kl}.

Naturally, the immittances so obtained are the values referred to the reference directions arbitrarily assigned to the boxes. If a single reference direction is changed, say that of box $\underline{k}$, then all the immittances with only one index equal to $\underline{k}$ are changed in sign.

That is, Z_{kl} and Y_{kl} $(k \neq l)$ change into $-Z_{kl}$ and $-Y_{kl}$, respectively, but Z_{kk} and Y_{kk} remain unchanged. This can easily be seen from the eqs. (14 & 16), or by means of the eqs. (4 & 6); or proved directly from the original eqs. (1 & 9), or from the definitions (5 & 13). When any number of the reference directions assigned to the boxes are changed, the corresponding changes in the signs of the (mutual-)immittances between the boxes can easily be found by changing one reference direction at a time and applying the above rule each time. It can be shown that this process is consistent.

As usual, a mutual-immittance between two boxes will be indicated in a figure by drawing an arc between them and placing the value of the immittance alongside. Of course, a system of arcs corresponding to mutual-impedances need not be the same as the system of arcs corresponding to the mutual-admittances (not even) for the same system of boxes; hence an indication as to the system of arcs used should always be given.

Problem 6. Assume that all the $\underline{N}$ boxes of a system of magnetically coupled boxes are connected in series in the same sense, as shown in Fig. 2. (There is no real limitation in this, since a reorientation of some of the boxes may always be made first, if necessary, with the corresponding changes made in the signs of the mutual-immittances.) Show that the impedance $\underline{Z}$, and the admittance $\underline{Y}$, of the series combination are given by:

$$Z = \sum_{k=1}^{N}\sum_{l=1}^{N} Z_{kl}\,, \qquad Y = 1/Z = \det(Y_{kl}) \Big/ \sum_{k=1}^{N}\sum_{l=1}^{N} \operatorname{cof} Y_{kl}\,. \qquad (17)$$

Write these out in full for the case of N=3 boxes. [HINT: The total complex voltage drop across the series combination is $V = V_1 + \ldots + V_N$, while all the currents are the same: $I_1 = I_2 = \ldots = I_N$. Then use eqs. (4).]

Fig. 2.

Problem 7. Assume that all the $\underline{N}$ boxes of a system of magnetically coupled boxes are connected in parallel in the same sense, as shown in Fig. 3. Show that the admittance $\underline{Y}$, and the impedance $\underline{Z}$, of the parallel combination are given by:

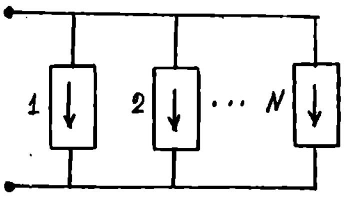


Fig. 3.

$$Y = \sum_{k=1}^{N}\sum_{l=1}^{N} Y_{kl}\,, \quad Z = \frac{1}{Y} = \det(Z_{kl}) \Big/ \sum_{k=1}^{N}\sum_{l=1}^{N} \operatorname{cof} Z_{kl}\,. \quad (1$$

Write these out in full for the case of 3 boxes. [HINT: The total complex current entering the parallel combination is $I = I_1 + I_2 + \ldots + I_N$, and all the voltage drops are the same: $V_1 = \ldots = V_N$. Then use eqs. (6 & 7).]

§3. NETWORKS WITH PASSIVE TWO-TERMINAL STRUCTURES.

We are now in a position to consider much more general two-rminal structures as the typical elements of a network, and in is way we may generalize all our previous results on networks f general series and parallel elements in the sinusoidal state.

First let us consider the series connection of a voltage source, current source, and an arbitrary two-terminal passive box (as shown n Fig. 1) as the typical element of a network. Any network may be nsidered as a network of such typical elements, since each of the sic elements may be considered as a particular case of such an ement. Only the terminals a and b of these generic (box-source) series conctions will be considered as the junction points or nodes of the twork; and all the internal nodes and meshes within the boxes will e ignored.

Fig. 1.

Now assume that we have a network (in the sinusoidal state at the angular frequency ω) of n_e such typical series elements, concted into n_n nodes in n_c components (= separate parts). Let us rbitrarily assign a reference direction to each of the typical series elements and number them consecutively from 1 to n_e. Next, let s omit exactly one node in each component and number the rest concutively from 1 to $n'_n = n_n - n_c$ = number of independent nodes. Also t us choose, in the usual way, a complete set of $n_m = n_e - n'_n$ indeendent meshes, oriented and numbered arbitrarily from 1 to n_m.

Let V_k, and I_k, denote the total complex voltage drop, and the mplex current, through the typical box-source series element k; and let V_{vs_k}, cs_k, and V_{box_k}, denote (respectively) the complex voltage drops rough the voltage source, the current source, and the box, in the ypical element k; all in the reference direction assigned to it. so, as usual, let E_k and D_k denote (respectively) the complex ltage rises (=EMF) through the voltage and current sources of the ypical element k, in the assigned reference directions. Then we ll have:

$$V_k = V_{vs_k} + V_{cs_k} + V_{box_k} = V_{box_k} - E_k - D_k \quad (k=1,2,\dots,n_e),$$

which the corresponding term should be omitted in any specific case in which a source or box is missing).

ut according to eqs. (4) of §2, the complex voltage drops across he boxes are:

$$V_{box_k} = \sum_{l=1}^{n_e} Z_{kl} I_l \qquad (k = 1,2,\dots,n_e),$$

here the Z_{kl} are the self- and mutual-impedances of the boxes. Thereore:

$$V_k = \sum_{l=1}^{n_e} Z_{kl} I_l - E_k - D_k \qquad (k = 1,2,\dots,n_e) \qquad (1)$$

aturally, if in a particular instance of the typical element shown n Fig.1, one of the things is missing, the corresponding terms in hese eqs.(1) should be understood to be absent.)

Besides, we have Kirchhoff's complex current and voltage laws for the independent nodes and meshes of the network, namely:

$$\sum_{k=1}^{n_e}(k,n)I_k = 0 \quad (n = 1,2,\ldots,n'_n) \quad \& \quad \sum_{k=1}^{n_e}[k,m]V_k = 0 \quad (m=1,2,\ldots,n_m). \quad (2$$

The $2n_e$ equations (1 & 2) are the general complex equations, in the $2n_e$ unknown complex currents and voltages, of the network in the sinusoidal state. They are exactly of the same form as the general complex equations (16, 17 & 18, of Ch. V, §1 p.125) of an arbitrary network of general series elements of the type used in Ch. V, §1, Fig. 2, p. 119. The only difference is that now the Z_{kl} are in general complex numbers (with real parts not necessarily zero) which are more complicated rational functions of ωi instead of special rational functions of the form $R+i\omega L+S/i\omega$ (or purely imaginary numbers of the form $i\omega L$ for $k \neq l$) as they were before (cf. Ch. V, §1, eqs. 5). However, concerning the formal treatment and methods of solution of these equations this is irrelevant.

In particular, the mesh method of treating the network (and the system of equations) may be introduced as usual by making the substitution (see Ch. VI):

$$I_k = \sum_{m=1}^{n_m}[k,m]J_m \qquad (k = 1,2,\ldots,n_e), \qquad (3)$$

of the complex currents, I_k, through the elements, in terms of the complex circulating currents, J_m, around the meshes $m (= 1,2,\ldots,n_m)$. Kirchhoff's complex current laws are then automatically satisfied; and by substituting (3) into (1) and the results into (2), we obtain (just as we did in Ch. VI, §2) the same familiar complex mesh equations of the network. Assuming that there are $n_{cs} (\leq n_m)$ current sources in the network, and that the typical elements with current sources are numbered first (consecutively from 1 to n_{cs}), the general complex canonical mesh equations of the network are:

$$\sum_{n=1}^{n_m} z_{mn}J_n = \widetilde{E}_m + \sum_{k=1}^{n_{cs}}[k,m]D_k = F_m \text{ (say)} \qquad (m=1,2,\ldots,n_m), \qquad (4a)$$

and

$$\sum_{m=1}^{n_m}[k,m]J_m = I_k \text{ (= known)} \qquad (k=1,2,\ldots,n_{cs}), \qquad (4b)$$

where the (known) mesh electromotiveforces of the voltage sources, $\widetilde{E}_m$, and the mesh (self- and mutual-)impedances z_{mn} are given as usual by:

$$\widetilde{E}_m = \sum_{k=1}^{n_e}[k,m]E_k \qquad (m = 1,2,\ldots,n_m) \qquad (5)$$

$$z_{mn} = \sum_{k=1}^{n_e}\sum_{l=1}^{n_e}[k,m]Z_{kl}[l,n] \qquad (m \,\&\, n = 1,2,\ldots,n_m). \qquad (6)$$

(The symbols z_{mn} are not used here for the same purposes as in §2!)

The practical rules given in Ch. VI, §2, pp. 166-7, for the de- rmination of the mesh impedances z_{mn} *in terms of the (now) box pedances* Z_{kl} *are also valid here too, of course; because they only pend on the form of eqs. (6), which is the same as that of eqs. (1), h. VI, §2; however, they will not be repeated here since there is o point in doing so. Nevertheless, one must notice that the mesh pedances are now more complicated rational functions of* ωi *and not st the simple rational functions (6) of Ch. VI, §2 (Prob. 2, p. 161).*

The general formulae (5) and (14 & 15) of Ch. VI, §4, for the lution of the complex canonical mesh equations of a network are lso valid here; because the equations solved there are formally e same as the ones we now have here; and so they will not be re- eated here. Also the treatment of the general existence and unique- ess theorems, given in Ch. VI, §5, is the same; but the definition of a non-singular network is the same only if the system of boxes is non-singular also; and concerning the eneral superposition principle, given in Ch. VI, §6, there is noth- ng more to be said.

Moreover, such concepts as the generalized natural frequencies f a network (see Prob. 3, Ch. VI, §4) are also the same and receive e same treatment, except for the form of the mesh impedances $z_{mn}(p)$ *the generalized frequency* $\underline{p}$*, which are now more complicated ration- l functions of* $\underline{p}$ *instead of the simple rational functions of the rm:* $r+pl+s/p$ *(with* $\underline{r}$, $\underline{l}$, $\underline{s}$ *= constants).*

<u>Problem 1</u>. Show that the complex equations of a network with- t (i.e., with null) mutual-impedances between the boxes are the following:

$$V_k = Z_k I_k - E_k - D_k \qquad (k = 1,2,\dots,n_e), \tag{7}$$

$$\sum_{k=1}^{n_e}(k,n)I_k = 0 \quad (n = 1,2,\dots,n_n'), \qquad \sum_{k=1}^{n_e}[k,m]\,V_k = 0 \quad (m = 1,2,\dots,n_m);$$

nd the complex canonical mesh equations are the same eqs. (4), but ow the mesh impedances are given by the simpler formula:

$$z_{mn} = \sum_{k=1}^{n_e}[k,m]\,Z_k\,[k,n] \qquad (m\ \&\ n = 1,2,\dots,n_m). \tag{8}$$

<u>Problem 2</u>. Show that the complex canonical mesh equations of network with no current sources ($n_{cs} = 0$) are the following:

$$\sum_{n=1}^{n_m} z_{mn}J_n = \tilde{E}_m = \sum_{k=1}^{n_e}[k,m]\,E_k \qquad (m = 1,2,\dots,n_m), \tag{9}$$

here the mesh impedances z_{mn} *are given by the same eqs. (6), or y eqs. (8) when there are no mutual-impedances between the boxes.*

<u>Example 1</u>. Consider the following so-called <u>Δ-Δ *3-phase net- ork*</u> *(shown in Fig. 2). Let us take: mesh 1=(123), mesh 2=(249$\bar{6}$), esh 3 = (5$\overline{763}$), mesh 4 = (789). Also, just for illustrative purposes,*

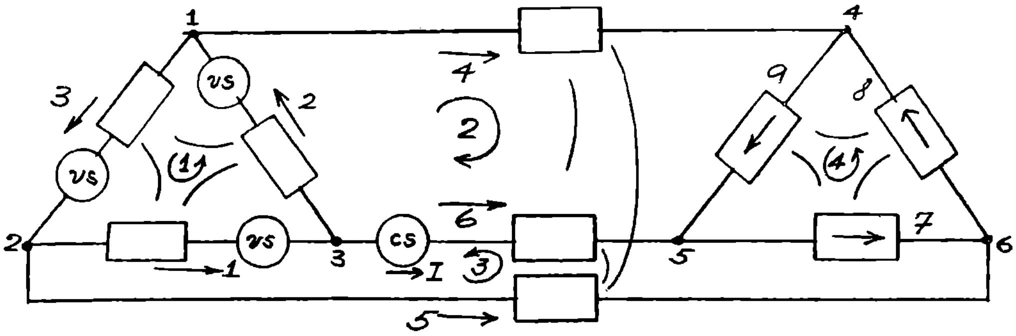


Fig. 2.

let us assume that there is a current source in the element 6, through which there is a (known) complex current $\underline{I}$ and an unknown complex voltage rise $\underline{D}$. The complex canonical mesh equations are:

$$\begin{aligned}
z_{11}J_1 + z_{12}J_2 + z_{13}J_3 + z_{14}J_4 &= E_1 + E_2 + E_3\\
z_{21}J_1 + z_{22}J_2 + z_{23}J_3 + z_{24}J_4 &= E_2 - D\\
z_{31}J_1 + z_{32}J_2 + z_{33}J_3 + z_{34}J_4 &= -E_1 - D\\
z_{41}J_1 + z_{42}J_2 + z_{43}J_3 + z_{44}J_4 &= 0\\
J_2 + J_3 &= -I
\end{aligned}$$

where:

$$\begin{aligned}
z_{11} &= Z_1 + Z_2 + Z_3 + 2(Z_{12} + Z_{23} + Z_{31})\\
z_{22} &= Z_2 + Z_4 + Z_9 + Z_6 - 2Z_{46}\\
z_{33} &= Z_1 + Z_5 + Z_7 + Z_6 - 2Z_{56}\\
z_{44} &= Z_7 + Z_8 + Z_9 + 2(Z_{78} + Z_{89} + Z_{97})\\
z_{12} &= z_{21} = Z_{12} + Z_2 + Z_{23}\\
z_{13} &= z_{31} = -Z_{12} - Z_1 - Z_{13}\\
z_{14} &= z_{41} = 0\\
z_{23} &= z_{32} = -Z_{12} + Z_{45} - Z_{46} - Z_{79} - Z_{56} - Z_6\\
z_{24} &= z_{42} = Z_9 + Z_{97} + Z_{98}\\
z_{34} &= z_{43} = -Z_7 - Z_{78} - Z_{79}.
\end{aligned}$$

Problem 3. Establish the complex canonical mesh equations for the following so-called Y-Y and Y-Δ connected 3 phase networks:

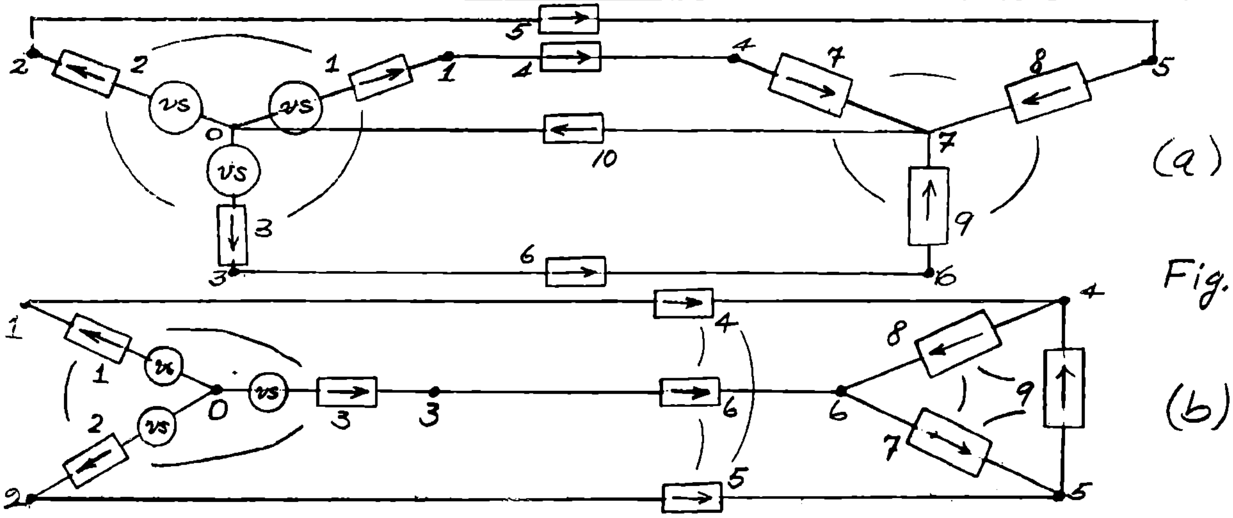


Fig. 3.

Problem 4. Considering the 3-phase networks shown in Figs. 2 & 3 (with the cs missing, and elements 8 and 9 interchanged, in Fig. 2), when: $Z_1=Z_2=Z_3$, $Z_{12}=Z_{23}=Z_{31}$, $Z_4=Z_5=Z_6$, $Z_{45}=Z_{56}=Z_{64}$, $Z_7=Z_8=Z_9$, $Z_{78}=Z_{89}=Z_{97}$, and $E_1=aE_2=a^2E_3$, where $a=\underline{/120^\circ}$ or $\underline{/-120^\circ}$, we say that they are ordinary balanced 3-phase networks. Assuming the networks to be non-singular, show that for these balanced 3-phase networks we shall have:

$$I_k = aI_{k+1} = a^2I_{k+2}\ , \qquad V_k = aV_{k+1} = a^2V_{k+2} \qquad (k = 1,4,7);$$

and for Figs. 2 & 3(a) we will then also have:

$$|I_4| = |I_5| = |I_6| = \sqrt{3}\,|I_7| = \sqrt{3}\,|I_8| = \sqrt{3}\,|I_9|$$

and for Figs. 3, using the double subscript notation:

$$|V_{12}| = |V_{23}| = |V_{31}| = \sqrt{3}\,|V_1| = \sqrt{3}\,|V_2| = \sqrt{3}\,|V_3|.$$

We will now show that any branch of elements connected in series may be considered as a single element, thereby obtaining a reduction in the number of elements of a network. (Cf. Prob. 19 of Ch. V, §1.) This will permit important simplifications in many networks. For this purpose, consider that in an arbitrary network of n_e (box-source series) elements, with the general complex equations (1 & 2), $\underline{N}$ of the elements are connected in series forming a branch. Without loss of generality we may assume these elements to be the elements $1,2,\ldots,N$, and that they are all oriented in the same sense along the branch. Denoting the common complex current through the elements of the branch by $\underline{I_b}$; we will then have for the total complex voltage drop along the branch (by eqs. 1):

$$\begin{aligned} V_b &= \sum_{k=1}^{N} V_k = \sum_{k=1}^{N}\Big(\sum_{l=1}^{N} Z_{kl}I_l + \sum_{l=N+1}^{n_e} Z_{kl}I_l - E_k - D_k\Big) \qquad (10a)\\ &= \Big(\sum_{k=1}^{N}\sum_{l=1}^{N} Z_{kl}\Big)I_b + \sum_{l=N+1}^{n_e}\Big(\sum_{k=1}^{N} Z_{kl}\Big)I_l - \sum_{k=1}^{N} E_k - \sum_{k=1}^{N} D_k\ ; \end{aligned}$$

and for the complex voltage drops across the other elements, we have:

$$V_k = \Big(\sum_{l=1}^{N} Z_{kl}\Big)I_b + \sum_{l=N+1}^{n_e} Z_{kl}I_l - E_k - D_k \qquad (k = N+1,\ldots,n_e). \qquad (10b)$$

Kirchhoff's complex current law equations for the $N-1$ intermediate nodes of the branch are already implied upon replacing I_1, $I_2,\ldots,I_N$ (all) by $\underline{I_b}$, and therefore drop out; and all the other current law equations remain the same. Moreover, V_b can be substituted for $\sum_{k=1}^{N} V_k$ in Kirchhoff's complex voltage laws, which otherwise remain the same.

Putting: $\quad E_b = \sum_{k=1}^{N} E_k\ , \quad D_b = \sum_{k=1}^{N} D_k\ , \quad Z_b = \sum_{k=1}^{N}\sum_{l=1}^{N} Z_{kl}\ ,$

and: (11)

$$Z_{bl} = \sum_{k=1}^{N} Z_{kl} \quad (l=N+1,\ldots,n_e), \qquad Z_{kb} = \sum_{l=1}^{N} Z_{kl} \quad (k=N+1,\ldots,n_e),$$

equations 10 (a & b) become:

$$\begin{aligned} V_b &= Z_bI_b + \sum_{l=N+1}^{n_e} Z_{bl}I_l - E_b - D_b \qquad (b = \text{branch}) \\ V_k &= Z_{kb}I_b + \sum_{l=N+1}^{n_e} Z_{kl}I_l - E_k - D_k \qquad (k = N+1, \dots, n_e). \end{aligned} \tag{12}$$

(Of course, since $Z_{kl} = Z_{lk}$ for all <u>k</u> & <u>l</u>, we will also have $Z_{bl} = Z_{lb}$ $(l=N+1,\dots,n_e)$.)

Comparing the above eqs. (12), together with the Kirchhoff's law equations left, with the general complex equations of an arbitrary network of n_e-N+1 similar elements, we see that they correspond to a network (with elements named: <u>b</u>, <u>N+1</u>, ..., <u>n_e</u>) in which the whole branch is a single element with the self-impedance Z_b, with mutual-impedances Z_{bl} (& Z_{kb}), with a voltage source of complex electro-motive force E_b, and with a current source (of complex current I_b) through which there is a complex voltage rise D_b, given by eqs. (11).

Notice that the value of Z_b is the same as that given by the first of eqs. (17), §2, so that the self-impedance of a branch is computed the same whether or not its elements are magnetically coupled with other elements not of the branch (but when they are coupled magnetically with other elements the admittance of the branch is not necessarily the inverse of the impedance, as in eqs. 17, §2, which is the difference). In words: The self-impedance of a branch is equal to the double (algebraic, if not all the elements of the branch are oriented in the same sense) sum of all the self- and mutual-impedances of and between the elements of the branch.

The (total) mutual-impedance of the branch with any other element not of the branch is equal to the (algebraic, idem.) sum of all the mutual-impedances of the elements of the branch with that other element.

When a network has several series branches, the above results may be used with advantage to reduce one branch at a time to a single element, thereby reducing the number of elements in the network. However, it must be noticed that when the mesh method is to be used, the advantage is only very slight.

<u>Example 2</u>. Consider the network shown in Fig. 4. All the boxes are passive. The three boxes 1, 2, 3, can be reduced to a single

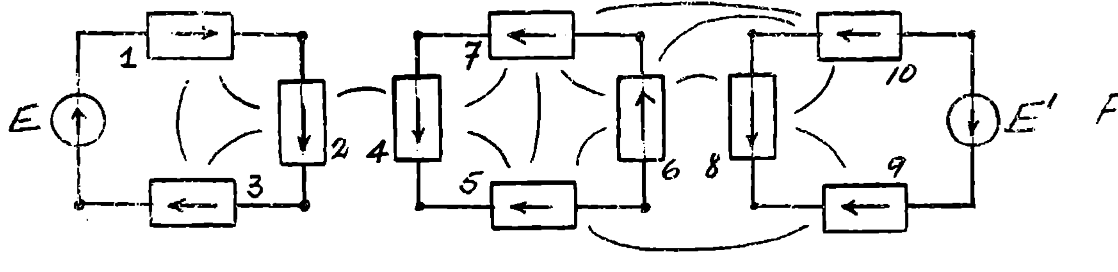


Fig. 4.

box with the self-impedance: $Z_a = Z_1+Z_2+Z_3+2(Z_{12}+Z_{23}+Z_{31})$, and a single non-vanishing mutual-impedance (that with element 4): $Z_{a4} = Z_{24}$, according to eqs. (11). Next the branch formed by the boxes 4, 5, 6, 7, may be reduced to a single box b. For this purpose it is convenient to imagine the reference direction assigned to box 5 reversed (with the corresponding changes in the mutual-impedances made, even if only mentally). By eqs. (11) we find for the self-impedance of this box: $Z_b = Z_4+Z_5+Z_6+Z_7+2(Z_{47}-Z_{45}-Z_{56}-Z_{57}+Z_{67})$, and for the non-vanishing mutual-impedances: $Z_{ba} = Z_{24}$, $Z_{b8} = Z_{68}$, $Z_{b9} = -Z_{59}$, and $Z_{b10} = Z_{6\,10} + Z_{7\,10}$. Finally, the branch formed by the boxes 8, 9, 10, can be reduced to a box c, with the self-impedance $Z_c = Z_8+Z_9+Z_{10} + 2(Z_{8\,10} - Z_{89})$, and a mutual-imedance with box b, $Z_{cb} = Z_{68}+Z_{59}+Z_{6\,10}+Z_{7\,10}$. With all this, the network becomes the one shown in Fig. 5.

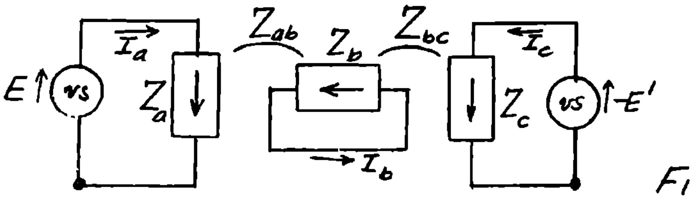


Fig. 5.

$$Z_a I_a + Z_{ab} I_b = E$$
$$Z_{ab} I_a + Z_b I_b + Z_{bc} I_c = 0$$
$$Z_{bc} I_b + Z_c I_c = -E'.$$

Once I_a, I_b, I_c, have been found, we will have: $I_1=I_2=I_3=I_a$, and: $I_4=-I_5=I_6=I_7=I_b$, and: $I_8=-I_9=I_{10}=I_c$; and with these complex currents, the complex voltage drops across the boxes 1, 2, ..., 10, can be found in the usual way (e.g., $V_5=Z_5I_5+Z_{45}I_4+Z_{56}I_6+Z_{57}I_7+Z_{59}I_9$).

§4. NETWORKS WITH TWO-TERMINAL PASSIVE STRUCTURES (cont'd).

Now let us consider the parallel connection of a voltage source, a current source, and an arbitrary two-terminal passive box (as shown in Fig. 1) as the typical unit of a network. Since each of the basic elements may be considered as a particular case of such typical box-source parallel elements, any network may be considered as a network of such elements. These box-source parallel elements are to be considered as units, and the internal meshes, as well as all the meshes and nodes within the boxes, are to be ignored; the treatment will be made in terms of the overall characteristics of the units.

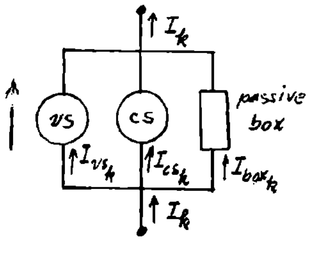


Fig. 1.

Consider a network (in the sinusoidal state at the angular frequency ω) of n_e such typical box-source parallel elements, connected into n_n nodes, in n_c components (=separate parts). Let us arbitrarily assign a reference direction to each of these typical units and number them consecutively from 1 to n_e. Next, let us omit exactly one node in each component (to be taken as base nodes for the corresponding component) and number the rest consecutively from 1 to $n_n' = n_n - n_c$ (the number of independent nodes). Also let us choose in the usual way a complete set of $n_m = n_e - n_n'$ independent meshes, arbitrarily oriented and numbered consecutively from 1 to n_m.

Let V_k be the complex voltage drop across, and I_k the complex terminal current entering (& leaving), the typical box-source parallel connected unit $\underline{k}$; and let I_{vs_k}, I_{cs_k}, I_{box_k}, be (respectively) the complex currents through the voltage source, the current source, and the box of the typical unit $\underline{k}$, all in the assigned reference direction. Also, as usual let E_k denote the complex *EMF.* of the *vs.* in element $\underline{k}$ in the assigned reference direction. Then we will have:

$$I_k = I_{vs_k} + I_{cs_k} + I_{box_k} \qquad (k = 1, 2, \ldots, n_e).$$

But according to eqs. (6) of §2, the complex currents through the boxes are:

$$I_{box_k} = \sum_{l=1}^{n_e} Y_{kl} V_l \qquad (k = 1, 2, \ldots, n_e),$$

where the Y_{kl} are the self- and mutual-admittances of the boxes, referred to the assigned reference directions. Therefore we will have:

$$I_k = I_{cs_k} + I_{vs_k} + \sum_{l=1}^{n_e} Y_{kl} V_l \qquad (k = 1, 2, \ldots, n_e). \qquad (1)$$

Of course, when a source or box is missing in a specific instance, the corresponding term in this equation should be considered as omitted.

Besides, we have Kirchhoff's complex current and voltage laws for the independent nodes and meshes of the network, namely:

$$\sum_{k=1}^{n_e} (k,n) I_k = 0 \quad (n = 1, 2, \ldots, n_n') \quad \& \quad \sum_{k=1}^{n_e} [k,m] V_k = 0 \quad (m = 1, 2, \ldots, n_m). \quad (2)$$

The $2n_e$ equations (1 & 2) are the *general complex equations* of the network in the sinusoidal state, in the $2n_e$ unknown complex currents and voltages. They are exactly of the same form as the general complex equations (6, 7, & 8, of Ch. V, §2, p. 145) of an arbitrary network of general parallel elements of the type used in Ch. V, §2, Fig. 2 (p. 139). The only difference is that now the Y_{kl} are in general complex numbers (with real parts not necessarily zero) which are more complicated rational functions of ωi instead of the special quantities (5) of Ch. V, §2, p. 144. Nevertheless, for a given frequency, these equations can be treated and solved as before.

In particular, the node method of treating the system of equations (1 & 2), and the network, can be introduced, as usual, by making the following substitution (cf. Ch. VII):

$$V_k = \sum_{n=1}^{n_n'} (k,n) U_n \qquad (k = 1, 2, \ldots, n_e) \qquad (3)$$

of the complex voltage drops V_k across the units, in terms of the complex nodal potentials U_n. Kirchhoff's complex voltage laws are then automatically satisfied; and by substituting (3) into (1) and the results into (2), we obtain (just as we did in Ch. VII, §1 & §2) the same familiar complex nodal equations of the network. Assuming that there are $n_{vs} (\leq n'_n)$ voltage sources in the network, and that the units with voltage sources are numbered first (consecutively from 1 to n_{vs}), the general complex canonical nodal equations are:

$$\sum_{n=1}^{n'_n} y_{mn} U_n = \tilde{I}_m - \sum_{k=1}^{n_{vs}} (k,m) I_{vs_k} \qquad (m = 1, 2, \ldots, n'_n), \qquad (4a)$$

and

$$\sum_{n=1}^{n'_n} (k,n) U_n = -E_k \qquad (k = 1, 2, \ldots, n_{vs}), \qquad (4b)$$

where the (total) complex currents $\tilde{I}_m$ entering the nodes $\underline{m}$ $(=1, \ldots, n'_n)$ through the current sources connected to them, and the nodal (self- and mutual-) admittances y_{mn} are given as usual by (cf. Ch. VII, §2):

$$\tilde{I}_m = -\sum_{k=1}^{n_e} (k,m) I_{cs_k} \qquad (m = 1, 2, \ldots, n'_n) \qquad (5)$$

$$y_{mn} = \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} (k,m) Y_{kl} (l,n) \qquad (m \& n = 1, 2, \ldots, n'_n). \qquad (6)$$

(The symbols y_{mn} are not used here for the same purposes as in §2!)

The practical rules (1º & 2º) given in Ch. VII, §2, pp. 199-200, for the determination of the nodal admittances y_{mn} in terms of the (now) box admittances Y_{kl} are also valid here, of course; because they only depend on the form of eqs. (6), which is the same as that of eqs. (1) of Ch. VII, §2. One must notice, however, that the nodal admittances are now more complicated rational functions of ωi and not necessarily of the simple form (5), Ch. VII, §2 (Prob. 3, p.194).

The general formulae (2) and (5 & 6) of Ch. VII, §4, for the solution of the complex canonical nodal equations of a network are also valid here; because the eqs. (4a & b) are formally the same as the eqs. (2), Ch. VII, §2, or as the eqs. (3) of Ch. VII, §4.

Also the treatment of the general existence and uniqueness theorems, as given in Ch. VII, §5, is the same here; but the definition of a non-singular network is the same only if the system of boxes of the network is non-singular also, in order that the definition be the same as when the basic elements within the boxes of the network are considered in detail as individuals (and not as grouped into boxes).

Concerning the general superposition principle, given in Ch. VII, §6, there is nothing more to be said.

The generalized natural frequencies $\underline{p}$ *of a network of box-source parallel elements are defined in the same way as in (the problem of) Ch. VII, §4; but one must notice that the admittances* $y_{mn}(p)$ *at the generalized natural frequency* $\underline{p}$ *are now more complicated rational functions of* $\underline{p}$*, instead of the simple rational functions of the form* $g+cp+\gamma/p$ *(with* g, c, γ = *constants).*

__Problem 1.__ Show that the complex equations of a network without (i.e., with null) mutual-admittances between the boxes are:

$$I_k = Y_k V_k + I_{cs_k} + I_{vs_k} \qquad (k = 1,2,\ldots,n_e),$$
$$\sum_{k=1}^{n_e}(k,n)I_k = 0 \quad (n=1,2,\ldots,n_n'), \qquad \sum_{k=1}^{n_e}[k,m]V_k = 0 \quad (m=1,2,\ldots,n_m); \tag{7}$$

and the complex canonical nodal equations are the same as eqs. (4), but now the nodal admittances are given by the simpler formula:

$$y_{mn} = \sum_{k=1}^{n_e}(k,m)Y_k(k,n) \qquad (m \,\&\, n = 1,2,\ldots,n_n'). \tag{8}$$

__Problem 2.__ Show that the complex canonical nodal equations of a network with no voltage sources ($n_{vs} = 0$) *are the following:*

$$\sum_{n=1}^{n_n} y_{mn}U_n = \tilde{I}_m = -\sum_{k=1}^{n_e}(k,m)I_{cs_k} \qquad (m = 1,2,\ldots,n_n') \tag{9}$$

where the nodal admittances y_{mn} *are given by the same eqs. (6), or by eqs. (8) when there are no mutual-admittances between the boxes.*

We will now show that any __parallel branch__ (i.e., elements connected in parallel), also called a __shunt branch__, may be considered as a single element. For this purpose, consider that in an arbitrary network of n_e *box-source parallel elements, with the general complex equations (1 & 2),* $\underline{N}$ *of the elements are connected in parallel, forming a (parallel) branch. Without loss of generality, we may assume these elements to be the elements 1,2,...,N, and that they are all oriented in the same sense. Denoting the common complex voltage drop across the elements of the parallel branch by* V_b*, the total complex terminal current entering the branch will be:*

$$\begin{aligned} I_b &= \sum_{k=1}^{N} I_k = \sum_{k=1}^{N}\Big(\sum_{l=1}^{N} Y_{kl}V_l + \sum_{l=N+1}^{n_e} Y_{kl}V_l + I_{cs_k} + I_{vs_k}\Big) \qquad (10a)\\ &= \Big(\sum_{k=1}^{N}\sum_{l=1}^{N} Y_{kl}\Big)V_b + \sum_{l=N+1}^{n_e}\Big(\sum_{k=1}^{N} Y_{kl}\Big)V_l + \sum_{k=1}^{N} I_{cs_k} + \sum_{k=1}^{N} I_{vs_k}\,; \end{aligned}$$

and for the complex terminal currents entering the other elements, we have:

$$I_k = \Big(\sum_{l=1}^{N} Y_{kl}\Big)V_b + \sum_{l=N+1}^{n_e} Y_{kl}\cdot V_l + I_{cs_k} + I_{vs_k} \quad (k=N+1,\ldots,n_e). \tag{10b}$$

Kirchhoff's complex current law equations remain the same, except for the substitution of I_b for $\sum_{k=1}^{N} I_k$. Moreover, Kirchhoff's complex voltage law is guaranteed for the intermediate meshes of the parallel branch by substituting V_b for each of $V_1, V_2, \ldots, V_N$; while all the rest of Kirchhoff's complex voltage law equations remain the same.

Putting:

$$I_{cs_b} = \sum_{k=1}^{N} I_{cs_k}, \qquad I_{vs_b} = \sum_{k=1}^{N} I_{vs_k}, \qquad Y_b = \sum_{k=1}^{N} \sum_{l=1}^{N} Y_{kl}, \tag{11}$$

and:

$$Y_{bl} = \sum_{k=1}^{N} Y_{kl} \quad (l = N+1, \ldots, n_e), \qquad Y_{kb} = \sum_{l=1}^{N} Y_{kl} \quad (k = N+1, \ldots, n_e),$$

equations 10(a & b) become:

$$\begin{aligned} I_b &= Y_b V_b + \sum_{l=N+1}^{n_e} Y_{bl} V_l + I_{cs_b} + I_{vs_b} && (b = \text{shunt branch}) \\ I_k &= Y_{kb} V_b + \sum_{l=N+1}^{n_e} Y_{kl} V_l + I_{cs_k} + I_{vs_k} && (k = N+1, \ldots, n_e). \\ & && (l = N+1, \ldots, n_e) \end{aligned} \tag{12}$$

(Of course, since $Y_{kl} = Y_{lk}$ for all k & l, we will also have $Y_{bl} = Y_{lb}$.)

Comparing the above eqs. (12), together with the Kirchhoff's law equations left, with the general complex equations of an arbitrary network of $n_e - N + 1$ box-source parallel elements, we see that they correspond to a network (with elements named: $b, N+1, \ldots, n_e$) in which the whole shunt branch is a single (similar) element with the self-admittance Y_b, with mutual-admittances Y_{bl} (& Y_{kb}), with a current source of complex current I_{cs_b}, and with a voltage source through which there is a complex current I_{vs_b}, given by eqs. (11).

Notice that the value of Y_b is the same as that given by the first of eqs. (18) of §2, so that the self-admittance of a shunt branch is computed the same whether or not its elements (boxes) are magnetically couped with other elements not of the shunt branch (the difference being that when they are magnetically couped with other elements the self-admittance of the shunt branch is not the inverse of the impedance necessarily, as in eqs. (18) of §2). In words, the self-admittance of a shunt branch is equal to the double sum (algebraic, if not all the elements of the shunt branch are oriented in the same sense) of all the self- and mutual-admittances of and between the elements (boxes) of the shunt branch.

Also, the (total) mutual-admittance of the shunt branch with any other element not of the branch is equal to the (algebraic) sum of all the mutual-admittances of the elements of the shunt branch with that other element.

When a network has several shunt branches, the above results may be used to reduce one shunt branch at a time to a single element.

Example. Consider the network shown in Fig. 2. All the boxes shown are passive. The shunt branch formed by the boxes 1, 2, 3, can be reduced immediately to a single box a, with the self-admittance $Y_a=Y_1+Y_2+Y_3$, and with only two non-vanishing mutual-admittances,

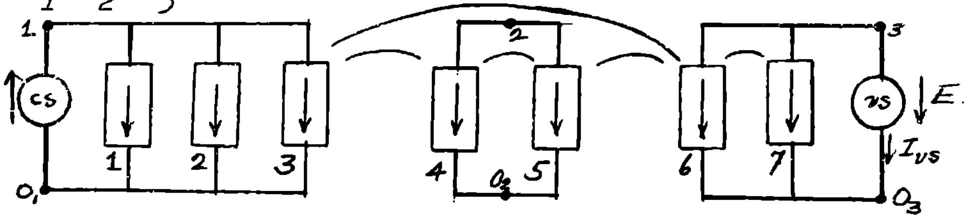


Fig. 2.

namely, those with boxes 4 & 6: $Y_{a4}=Y_{34}$, $Y_{a6}=Y_{36}$, according to eqs. (11). Next, the elements 4 & 5 may be considered as a shunt branch, which may be reduced to a single box b, with the self-admittance $Y_b=Y_4+Y_5+2Y_{45}$, and with mutual-admittances: $Y_{ba}=Y_{43}$ and $Y_{b6}=Y_{56}$. Finally, the shunt branch formed by the boxes 6 & 7 may be reduced to a single box c, with the self-admittance: $Y_c=Y_6+Y_7+2Y_{67}$, and with the mutual-admittances: $Y_{ca}=Y_{63}$ and $Y_{cb}=Y_{65}$. After all these reductions, the network takes the simple form shown in Fig. 3, with the equations:

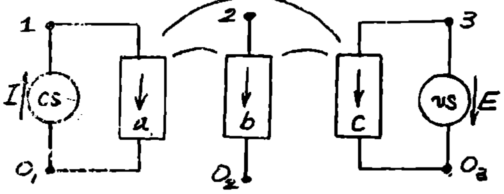


Fig. 3.

$$Y_a V_a + Y_{ab} V_b - Y_{ac} E = I$$
$$Y_{ba} V_a + Y_b V_b - Y_{bc} E = 0$$
$$Y_{ca} V_a + Y_{cb} V_b - Y_c E = -I_{vs}.$$

In these equations, I and E are known. From the first two equations we can find $V_a=V_1=V_2=V_3$ and $V_b=V_4=V_5$; whereas: $V_6=V_7=-E$ were already known. Then from the last equation we can find I_{vs}. The complex currents through the boxes can be found in terms of the complex voltage drops as usual (e.g., $I_4=Y_{43}V_3+Y_4V_4+Y_{45}V_5=-I_5$).

§5. THE CASE OF ACTIVE TWO-TERMINAL STRUCTURES.

Consider an arbitrary system of N (perhaps) active magnetically coupled two-terminal structures (active boxes), numbered 1, 2, ..., N, as shown in Fig. 1.[*] (When N=1, we will have the case of a single box.) Assume that all the sources within the boxes are sinusoidal of a same given frequency. Assume also that the terminals of each box are mutually accessible through elements within the corresponding box, and that a reference direction is arbitrarily assigned to each box, from one of its terminals to the other.

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[*] We assume that at least one of the boxes of a system (to be called) of active boxes is active.

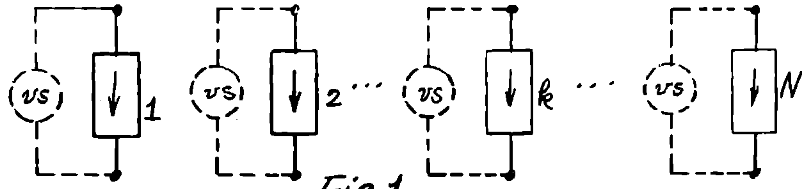


Fig.1

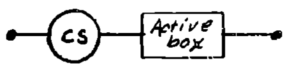


Fig.2.

Suppose that we now connect sinusoidal voltage sources of the same given frequency across the terminals of each box, as shown by the dotted lines in Fig. 1. When this network is non-singular we will say that the given <u>system of active boxes is non-singular</u> (at the given frequency). Let V_k and I_k denote (respectively) the complex voltage drop and current through the box <u>k</u>, in the assigned reference direction. Let I_k° denote the complex current through the box <u>k</u> (in the assigned reference direction) due to the totality of sources within the boxes, with the voltage sources outside the boxes nullified (i.e., replaced by short-circuits), and let I_k' denote the complex current through the box <u>k</u> (in the assigned reference direction) due to the voltage sources outside the boxes, with all the sources within the boxes nullified (cf. Ch. VI, §6, p. 182). Then, according to the superposition principle, we shall have $I_k = I_k' + I_k^\circ$. But when all the sources within the boxes are nullified, we shall have an ordinary system of magnetically coupled passive boxes (unless a box of the system is of the form shown in Fig. 2; in which case this box may in a sense be considered as a current source that may be incorporated into one of the other boxes, thus reducing the number of boxes in the system), for which the results of §2 apply. Thus we will have (by eqs. 6, §2):

$$I_k' = \sum_{l=1}^{N} Y_{kl} V_l \qquad (k = 1, 2, \ldots, N), \qquad (1)$$

where the Y_{kl} are the (self- and mutual-) <u>admittances of the passified</u> (or passivized) <u>boxes</u> (when all the sources within the active boxes are nullified). Consequently, we shall have (substituting in $I_k = I_k' + I_k^\circ$):

$$I_k = \sum_{l=1}^{N} Y_{kl} V_l + I_k^\circ \qquad (k = 1, 2, \ldots, N). \qquad (2)$$

Now assume that our system of active boxes is part of a network in the sinusoidal state at the given frequency. We can be sure that the complex voltage drops, V_k, and complex currents, I_k, through the boxes of the system shall still be related by eqs. (2), unless there are boxes not of the system that are magnetically coupled with them (i.e., as long as the system is magnetically closed). Because, if the system of boxes were to be disconnected from the network and voltage sources of complex EMF. V_k applied to them, we could be sure that the actual complex currents and voltage drops through the elements of the boxes in the given network would satisfy the system of equations of this new network also; and since it is non-singular, they would be the only possible ones. Therefore the I_k would be the complex currents through the boxes in the new network also and hence they would satisfy eqs. (2). Thus no matter how the active boxes are connected with other elements, the complex currents and voltage drops through them shall always be related by eqs. (2), as long as they form a magnetically closed non-singular system of (active) boxes.

Eqs. (2) can be solved for the complex voltage drops V_k in terms of the complex currents I_k through the active boxes, as follows:

$$V_k = \sum_{l=1}^{N} Z_{kl} I_l + V_k^\circ \qquad (k = 1, 2, \ldots, N), \tag{3}$$

where:

$$Z_{kl} = \operatorname{cof} Y_{lk} / \det(Y_{kl}) \qquad (K \,\&\, l = 1, 2, \ldots, N), \tag{4}$$

and:

$$V_k^\circ = - \sum_{l=1}^{N} Z_{kl} I_l^\circ \qquad (k = 1, 2, \ldots, N). \tag{5}$$

The Z_{kl} are called the (self- and mutual-) impedances of the passified boxes. The independent terms, I_k°, in eqs. (2) are called the short-circuit complex currents of the active boxes; because they are the complex currents through them when all the $V_k=0$, as can be seen from eqs. (2), i.e., when all the boxes are short-circuited. Similarly, the independent terms V_k° in eqs. (3) are called the open-circuit complex voltage drops through the active boxes of the given system; because they are the complex voltage drops across them when all the $I_k=0$, as can be seen from eqs. (3), i.e., when all the boxes are opened. It will be found convenient to use: $E_k^\circ = -V_k^\circ$ for the open-circuit complex voltage rises through the active boxes of the system $(k=1,2,\cdots,N)$.

The equations corresponding to a single magnetically isolated active non-singular two-terminal structure are the following:

$$\underset{=ZI-E^\circ}{V=ZI+V^\circ}, \qquad I=YV+I^\circ, \quad \text{where} \quad \underset{=ZE^\circ}{V^\circ=-ZI^\circ}, \qquad I^\circ=-YV^\circ=YE^\circ. \tag{6}$$

It may be in place to mention that these equations contain the essence of the classical theorems of Thévenin (1883) and Norton (later), which seem to have been known to H. Helmholtz since 1853.

One must notice the important difference between the eqs. (2 & 3), of this section, for a system of active boxes, and the corresponding eqs. (6 & 4) of §2 for a system of passive boxes. When the boxes of a system are active the relations between the complex voltage drops and currents through the boxes are no longer homogeneous; but instead they are general linear relationships, with independent terms due to the sources within the boxes of the system.

The methods that can be used to measure experimentally the characteristic quantities Y_{kl}, Z_{kl}, I°_k, and V°_k, of a system of active boxes should now be clear to the careful reader, by this time. It might perhaps be in place only to mention that each of these quantities should be measured in the presence of all the boxes.

Problem 1. Show that: $Y_{kl} = \operatorname{cof} Z_{lk}/\det(Z_{kl})$ and $I^\circ_k = -\sum_{l=1}^{N} Y_{kl}V^\circ_l$. (7)

Problem 2. Obtain eqs. (3) directly by assuming sinusoidal current sources (of the given frequency) to be connected across the boxes instead of the voltage sources used above, as shown in Fig. 1.

Problem 3. Suppose that our system of N active boxes is part of a network in the sinusoidal state at the given frequency. Let V'_k and I'_k be the complex voltage drops and currents through the N boxes, due to all the sources of the network outside the N boxes, when all the sources within them are nullified; and let V''_k and I''_k be the complex voltage drops and currents through the boxes, due to the sources within the boxes of the system, when all the sources of the network outside them are nullified. Using the superposition principle and the results of §2 for passive boxes, show that we shall have for the complex voltage drops V_k and currents I_k through the boxes in the network:

$$V_k = V'_k + V''_k, \qquad I_k = I'_k + I''_k, \quad \text{and:} \quad V'_k = \sum_{l=1}^{N} Z_{kl}I'_l, \qquad I'_k = \sum_{l=1}^{N} Y_{kl}V'_l, \tag{8}$$
$$V^\circ_k = V_k - \sum_{l=1}^{N} Z_{kl}I_l = V''_k - \sum_{l=1}^{N} Z_{kl}I''_l, \qquad I^\circ_k = I_k - \sum_{l=1}^{N} Y_{kl}V_l = I''_k - \sum_{l=1}^{N} Y_{kl}V''_l.$$

Problem 4. Show that when all the N active boxes of a magnetically closed system are connected in series (in the same sense), the series combination will be an active box (with the eqs. 6) with a passified impedance Z, and an open-circuit complex voltage drop V°, given by:

$$Z = \sum_{k=1}^{N}\sum_{l=1}^{N} Z_{kl} \qquad \text{and} \qquad V^\circ = \sum_{k=1}^{N} V^\circ_k. \tag{9}$$

Problem 5. Show that when all the N active boxes of a (magnetically closed) system are connected in parallel (in the same sense), the parallel combination will be an active box (with the eqs. 6) with a passified admittance Y and a short-circuit complex current I°, given by:

$$Y = \sum_{k=1}^{N}\sum_{l=1}^{N} Y_{kl} \qquad \text{and} \qquad I^\circ = \sum_{k=1}^{N} I^\circ_k. \tag{10}$$

Problem 6. Show that if an active magnetically isolated box (with the eqs. 5) is connected across a magnetically isolated passive box of impedance Z', and admittance $Y'=1/Z'$, as shown in Fig. 3,

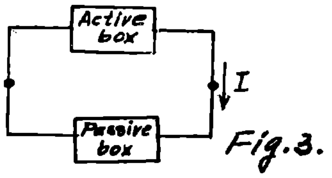


Fig.3.

we shall have: $I = \frac{-V°}{Z+Z'}$ and $V = \frac{-I°}{Y+Y'}$. (11)
These equations express (Helmholtz's-) Thévenin's and Norton's theorems (respectively) in their more common form. (Hint: $V=ZI+V°=-Z'I$ and $I=YV+I°=-Y'V$.)

We will now consider networks of active two-terminal structures. For this purpose we will consider structures of the form shown in Fig. 4, or of the form shown in Fig. 5, as the typical units of the network; these will now be called simply the (series, and shunt or parallel, resp.) elements of a network. It shall be found convenient to assume that the active box in the typical series element is not, in itself, of the same form of a current source in series with another box; and that the active box in the typical parallel element is not, in itself, a parallel connection of a voltage source with another box. However, there is no point in adding a voltage source in the typical series element, because it could be incorporated into the active box without any inconvenience; and, similarly, there is no point in adding a current source in the typical parallel element, because it could be incorporated into the active box without any inconvenience.

Consider a sinusoidal current network of n_e typical active series elements (of the type shown in Fig. 4) connected into n_n nodes in n_c components (=separate parts), with $n_n' = n_n - n_c$ independent nodes and $n_m = n_e - n_n'$ independent meshes (ignoring, of course, all the nodes and meshes within the boxes). As usual, let all the typical elements, independent nodes, and independent meshes, be arbitrarily numbered and oriented. Denoting the (total) complex voltage drops and currents through the series elements of the network by V_k and I_k (resp.), and the complex voltage rises through the "explicit" current sources by D_k, the complex equations of the network will be (according to eqs. 3), assuming that the active boxes in the n_e elements form a non-singular system,:

$$V_k = V_{box_k} + V_{cs_k} = \sum_{l=1}^{n_e} Z_{kl} I_l + V_k^\circ - D_k \quad (k = 1, 2, \ldots, n_e),$$
$$\sum_{k=1}^{n_e} (k,n) I_k = 0 \quad (n=1,2,\ldots,n_n'), \qquad \sum_{k=1}^{n_e} [k,m] V_k = 0 \quad (m=1,2,\ldots,n_m). \tag{12}$$

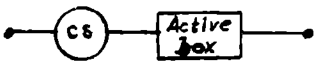


Fig.4

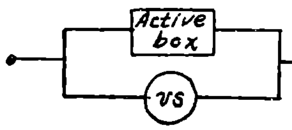


Fig.5.

We emphasize that the form chosen for the generic series element, shown in Fig. 4, does not mean that every particular instance of a series element in a specific network has to have a current source and an active box. Rather, it should be understood that when in a particular instance the current source or the active box, or the sources within the box, are absent, then the corresponding terms in the first of eqs. (12) should be considered as omitted. And similar remarks should be understood about the generic parallel element shown in Fig. 5, networks of which will be considered below.

Comparing eqs. (12) with the general equations (1 & 2 of §3) of a network of passive box-source series elements (of the form shown in Fig. 1, §3), we see that a non-singular network of active boxes can always be reduced to an equivalent network with passive boxes (equivalent in the sense that both networks have the same system of equations), the passive boxes of the latter being the passified boxes of the former, and the complex electromotive forces of the voltage sources in the latter being the negatives, $-V_k^\circ$, of the open-circuit complex voltage drops (i.e., the open-circuit complex voltage rises: $-V_k^\circ = E_k^\circ$, say) through the active boxes in the former (in the presence of all the boxes).

In particular, we see that the active box in the element k can be replaced by a voltage source of complex electromotive force E_k° in series with a passive box of self-impedance $Z_k = Z_{kk}$ and mutual impedances Z_{kl} (the self- and mutual-impedances of the passified boxes).

In general, each E_k° may depend on all the sources within all the active boxes of the network; but if a particular box, k' (say), is magnetically isolated, the open-circuit complex voltage rise through it: $E_{k'}^\circ = -V_{k'}^\circ = Z_{k'} I_{k'}$, and its impedance $Z_{k'}$ when all the sources within it are nullified, shall be the same as when the box is isolated, and therefore will depend only on the box k'; and this is in effect the well-known form of (Helmholtz's) Thévenin's theorem. (Cf. Ch. IX, §4.) Thus we see that in the preceding statement we have a substantial generaliztion of the theorem of (Helmholtz and) Thévenin. (We might add that Thévenin's theorem is stated wrongly in the book on Electric Circuits (p. 469) written by the M.I.T. electrical engineering staff, inasmuch that it calls for the passive box to have the impedance of the active box when all its sources are shorted, and not nullified. This, in turn, would be correct if the active box had voltage sources within it only.)

With the above reduction of any given network of active structures of the form shown in Fig. 4 to a network of passive box-source series elements (of the form shown in Fig. 1, §3), that were treated in detail in §3, we may consider this part of the topic on networks of active structures as concluded.

Problem 7. By making the substitution: $I_k = \sum_{m=1}^{n_m} [k,m] J_m \ (k=1,\ldots,n_e)$ in eqs. (12), establish the following complex canonical mesh equations of a network of n_e active series elements:

$$\begin{aligned} \sum_{n=1}^{n_m} z_{mn} J_n &= \tilde{E}_m^\circ + \sum_{k=1}^{n_{cs}} [k,m] D_k \qquad (m = 1,2,\ldots,n_m), \\ \sum_{n=1}^{n_m} [k,n] J_n &= I_k \qquad (k = 1,2,\ldots,n_{cs}), \end{aligned} \tag{13}$$

where the mesh impedances z_{mn}, and the mesh electromotive forces $\tilde{E}_m^\circ$ (due to the sources within the active boxes), are given in terms of the passified impedances Z_{kl}, and the open-circuit complex voltages $V_k^\circ = -E_k^\circ$, by (for $m \& n = 1, 2, \ldots, n_m$):

$$z_{mn} = \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} [k,m] Z_{kl} [l,n], \qquad \tilde{E}_m^\circ = \sum_{k=1}^{n_e} [k,m] E_k^\circ = -\sum_{k=1}^{n_e} [k,m] V_k^\circ, \tag{14}$$

assuming that there are $n_{cs} (\leq n_m)$ elements with "explicit" current sources, numbered first. Also establish the solution formulae of these canonical equations. (See eqs. 14 & 15 of Ch. VI, §4.)

Let us now consider a sinusoidal current network of n_e typical active parallel elements (of the type shown in Fig. 5) connected into n_n nodes in n_c components, with $n'_n = n_n - n_c$ independent nodes and $n_m = n_e - n'_n$ independent meshes (ignoring, of course, all the nodes and meshes within the boxes). Let all the typical elements, independent nodes, and independent meshes, be arbitrarily numbered and oriented. Denoting the (total) complex voltage drops and currents through the parallel elements of the network by V_k and I_k (resp., $k=1,\ldots,n_e$) and the complex currents through the "explicit" voltage sources by I_{vs_k}, the complex equations of the network will be (assuming that the active boxes in the n_e elements form a non-singular system) by eqs. (2):

$$\begin{gathered} I_k = I_{box_k} + I_{vs_k} = \sum_{l=1}^{n_e} Y_{kl} V_l + I_k^\circ + I_{vs_k} \qquad (k = 1,2,\ldots,n_e), \\ \sum_{k=1}^{n_e} (k,n) I_k = 0 \quad (n=1,2,\ldots,n'_n), \qquad \sum_{k=1}^{n_e} [k,m] V_k = 0 \quad (m=1,2,\ldots,n_m). \end{gathered} \tag{15}$$

Comparing these equations with the general equations (1 & 2, §4) of a network of passive box-source parallel elements (of the form shown in Fig. 1, §4), we see that a non-singular network of active boxes can always be reduced to an equivalent network with passive boxes (with the same system of equations), the passive boxes in the latter being the passified boxes of the former, and the complex currents through the current sources in the latter being the short-circuit complex currents through the active boxes of the former (with all the boxes present).

In particular, we see that the generic active box in the element k can be replaced by a current source of complex current I°_k in parallel (shunted) with a passive box of self-admittance $Y_k = Y_{kk}$ and mutual-admittances Y_{kl} (the self- and mutual-admittances of the passified boxes). This is a substantial generalization of the theorem of (Helmholtz and) Norton. ¶In general, each I°_k shall depend on all the boxes of the network, so that one should imagine the system of active boxes to be given before any box is replaced by a current source shunted by a passive box; but if a particular box, k'(say), is magnetically isolated, the short-circuit complex current through it: $I^\circ_{k'} = -Y_{k'}V^\circ_{k'}$, and the admittance $Y_{k'}$ of the box when all the sources within it are nullified, will be independent of the other boxes; and the statement, that in this particular case the magnetically isolated active box can be replaced by a current source of complex current equal to the short-circuit complex current $I^\circ_{k'}$ of the active box in shunt with the passified box of admittance $Y_{k'}$, is the better known form of (Helmholtz's-) Norton's theorem. (Cf. Ch. IX, §5.)

With the above reduction of any given network of active structures of the form shown in Fig. 5 to a network of passive box-source parallel elements (of the form shown in Fig. 1, §4), which were considered in detail in §4, we shall consider the topic on networks of active two-terminal structures concluded.

Problem 8. By making the substitution: $V_k = \sum_{n=1}^{n'_n}(k,n)U_n$ (for $k=1,2,\ldots,n_e$) in eqs. (15), establish the following complex canonical nodal equations of a network of n_e active parallel elements:

$$\sum_{n=1}^{n'_n} y_{mn}U_n = \tilde{I}^\circ_m - \sum_{k=1}^{n_{vs}}(k,m)I_{vs_k} \qquad (m = 1, 2, \ldots, n'_n)$$

$$\sum_{n=1}^{n'_n}(k,n)U_n = -E_k \ \text{(the known EMF of } vs_k\text{)} \qquad (k=1,\ldots,n_{vs}) \tag{16}$$

where the nodal admittances y_{mn}, and the nodal complex currents $\tilde{I}^\circ_m$ (due to the sources within the active boxes), are given in terms of the passified admittances Y_{kl}, and the short-circuit complex currents I°_k, by (for $m \,\&\, n = 1,2,\ldots,n'_n$):

$$y_{mn} = \sum_{k=1}^{n_e}\sum_{l=1}^{n_e}(k,m)Y_{kl}(l,n), \qquad \tilde{I}^\circ_m = -\sum_{k=1}^{n_e}(k,m)I^\circ_k\,, \tag{17}$$

assuming that there are $n_{vs}(\leq n'_n)$ typical active shunt elements with "explicit" voltage sources (of complex EMFs. E_k), numbered first. Also establish the solution formulae for these canonical equations. (Cf. eqs. 5 & 6 of Ch. VII, §4.)

CHAPTER IX: THE CLASSICAL NETWORK THEOREMS.

In this chapter we wish to present some useful general (now classical) network theorems, which are mentioned briefly and used here and there in the literature, but which are too often presented loosely (and many times mistakenly), and hardly ever in a general form. (The superposition principle, which is the most important of these general results, has already been presented in detail in chapters VI & VII, and so will be omitted here.)

It will be convenient to lay down the meaning of a term to be used in the following. By the effects of something (on the currents and voltages) in a network (in the sinusoidal state, here), we shall understand the (algebraic) increments in the (complex) currents and voltages in the network that are produced by that thing.

§ 1. THE RECIPROCITY THEOREM.

In a non-singular sinusoidal current network with no current sources, the effect of a voltage source inserted in the branch (= passive box-source series element) k on the current through the branch l is the same as the effect of this voltage source (removed from the branch k and) inserted in the branch l on the current through the branch k.

To prove this, consider the complex canonical mesh equations of the network, namely (eqs. 9 of Ch. VIII, §3):

$$\sum_{n=1}^{n_m} z_{mn} J_n = \widetilde{E}_m = \sum_{k=1}^{n_b} [k,m] E_k \qquad (m=1,2,\ldots,n_m) \qquad (1)$$

where the mesh impedances z_{mn} are given in terms of the box impedances by eqs. (6) of Ch. VIII,§3. The solution of eqs. (1) is:

$$J_n = \sum_{m=1}^{n_m} \widetilde{E}_m \operatorname{cof} z_{mn} / \det (z_{mn}) = \sum_{m=1}^{n_m} y_{nm} \widetilde{E}_m \quad (n=1,2,\ldots,n_m). \qquad (2)$$

The quatities $y_{nm} = \operatorname{cof} z_{mn} / \det (z_{mn})$, which are of the nature of admittances, are called the (short-circuit) mesh admittances of the network, and $\widetilde{E}_m$ is called the (total) exciting or driving complex electromotive force (through the voltage sources) of the mesh m. y_{mm} is called the driving-point (or self) admittance of the mesh m, and y_{mn} $(m \neq n)$ is called the (mutual or) transfer admittance between the meshes m & n (or from the mesh n to the mesh m). The frequently used reciprocals of these quantities are called the driving-point and transfer (mesh) impedances, respectively.

Now since the box impedances are symmetric (i.e., $Z_{kl} = Z_{lk}$ for all $\underline{k}$ & $\underline{l}$; see Prob. 2, §2, Ch. VIII), it is easy to see from eqs. (6) of Ch. VIII, §3 (see Ch. VI, §2, Prob. 1) that the mesh impedances z_{mn} are also symmetric (i.e., $z_{mn}=z_{nm}$ for all the meshes m & $n=1,\ldots,n_m$) and so we shall also have that all the cofactors and the determinant of the z_{mn} are also symmetric; hence the mesh admittances y_{mn} shall also be symmetric (i.e., $y_{mn}=y_{nm}$, for m & $n=1,2,\ldots,n_m$). As a consequence, if all the voltage sources of the network are nullified (shorted) except a single voltage source of complex electromotive force E in element $\underline{k}$ (or if there is a single voltage source in the network, in element $\underline{k}$), the complex current through the element $\underline{l}$ (=effect of the voltage source in the element $\underline{k}$), namely:

$$I_l=\sum_{n=1}^{n_m}[l,n]J_n=\sum_{n=1}^{n_m}\sum_{m=1}^{n_m}[l,n]\,y_{nm}\widetilde{E}_m=\sum_{n=1}^{n_m}\sum_{m=1}^{n_m}[l,n]\,y_{nm}\sum_{k=1}^{n_e}[k,m]E_k$$
$$=\sum_{n=1}^{n_m}\sum_{m=1}^{n_m}[l,n]\,y_{nm}\,[k,m]E\ , \qquad (3)$$

will be equal to:

$$I_k=\sum_{m=1}^{n_m}\sum_{n=1}^{n_m}[k,m]\,y_{mn}\,[l,n]E\ , \qquad (3a)$$

which is the complex current through the element $\underline{k}$ when the only voltage source (left) in the network is changed to (and connected in series with) the element $\underline{l}$; which proves the reciprocity theorem.

Another form of <u>reciprocity</u> <u>theorem</u> can be obtained by considering an arbitrary sinusoidal current (non-singular) network with passive boxes and without voltage sources, the complex canonical nodal equations of which (considered as a network of box-source shunt elements) are (see eqs. 9 of Ch. VIII, §4):

$$\sum_{n=1}^{n_n'}y_{mn}U_n=\widetilde{I}_m=-\sum_{k=1}^{n_e}(k,m)I_{cs_k} \qquad (m=1,2,\ldots,n_n'), \qquad (4)$$

where the nodal admittances y_{mn} are given in terms of the box admittances by eqs. (6) of Ch. VIII, §4. The solution of eqs. (4) is:

$$U_n=\sum_{m=1}^{n_n'}\widetilde{I}_m\operatorname{cof}y_{mn}/\det(y_{mn})=\sum_{m=1}^{n_n'}z_{nm}I_m \qquad (n=1,2,\ldots,n_n'). \qquad (5)$$

(The y_{mn} and z_{mn} used here do <u>not</u> have the same meaning as they had above in the previous treatment of the reciprocity theorem!)

The quantities $\widetilde{I}_m$ are the (total) <u>exciting</u> or <u>driving</u> <u>complex</u> <u>currents</u> arriving to the nodes $\underline{m}$ through the current sources connected to them. The quantities $z_{nm}=\operatorname{cof}y_{mn}/\det(y_{mn})$, which are of the nature of impedances, are called the (<u>open-circuit</u>) <u>nodal</u> <u>impedances</u>; z_{nn} is called the driving-point (or <u>self-)impedance</u> <u>of</u> <u>the</u> <u>node</u> $\underline{n}$, and z_{mn} ($m\neq n$) is called the <u>mutual</u> or <u>transfer</u> <u>impedance</u> between the nodes $\underline{m}$ & $\underline{n}$ (or from the node $\underline{n}$ to the node $\underline{m}$). The frequently used reciprocals of these quantities are called the <u>driving-point</u> and <u>transfer</u> (nodal) <u>admittances</u>, respectively.

As the box admittances are symmetric (see Prob. 2, Ch. VIII, §2), it can easily be shown (as above) that the nodal admittances are also symmetric, and then that the transfer impedances are symmetric. Consequently, when all the current sources of the network are nullified (opened) except a single current source of complex current I shunted in element k (in order to find its effects), or if there is a single current source in the network, shunted in element k, the complex voltage drop through the element (box) l, namely:

$$V_l=\sum_{n=1}^{n'_n}(l,n)U_n=\sum_{n=1}^{n'_n}\sum_{m=1}^{n'_n}(l,n)z_{nm}\tilde{I}_m=\sum_{n=1}^{n'_n}\sum_{m=1}^{n'_n}(l,n)z_{nm}\Big(-\sum_{k=1}^{n_e}(k,m)I_{cs_k}\Big)=$$
$$=-\sum_{n=1}^{n'_n}\sum_{m=1}^{n'_n}(l,n)z_{nm}(k,m)I \tag{6}$$

will be the same as:

$$V_k=-\sum_{m=1}^{n'_n}\sum_{n=1}^{n'_n}(l,m)z_{mn}(l,n)I \tag{6a}$$

which is the complex voltage drop across the element (box) k when the only current source (left) in the network is changed to the element (box) l (shunted across it).

That is, the effect produced in the voltage drop across the box k by a current source connected across the box l, is the same as the effect it would produce in the voltage drop across the box l were it to be connected, instead, across the box k.

§ 2. THE COMPENSATION THEOREM.

The effects of a change ΔZ in the self-impedance (only) of an element of an arbitrary network in the sinusoidal state (assumed to be and remain non-singular) are the same as the effects of a compensating voltage source inserted in (series in) the modified branch (as changed) of complex electromotive force $-\Delta Z\cdot I$ in the reference direction of the element, where I is the complex current ~~I~~ through the changed branch before the modification, after the network resumes the sinusoidal state.

Actually we are going to prove much more than this, the above statement being a particular case of our result. To do this, consider the given network as a network of n_e passive box-source series elements with the complex equations (1 & 2) of Ch. VIII, §3, namely:

$$V_k=\sum_{l=1}^{n_e}Z_{kl}I_l-E_k-D_k \qquad (k=1,2,\dots,n_e),$$
$$\sum_{k=1}^{n_e}(k,n)I_k=0 \quad (n=1,2,\dots,\overset{\leq 2n_e}{n'_n}), \qquad \sum_{k=1}^{n_e}[k,m]\,V_k=0 \quad (m=1,2,\dots,\overset{=n_e-n'_n}{n_m}). \tag{1}$$

Now consider the network after the self- and mutual-impedances of the box in the element h (only) are changed and the sinusoidal state resumed. Denoting the new quantities in this network with dashes (primes), the complex equations of the modified network will be:

$$V'_k = \sum_{l=1}^{n_e} Z'_{kl} I'_l - E_k - D'_k \qquad (k = 1, 2, \ldots, n_e),$$
$$\sum_{k=1}^{n_e} (k,n) I'_k = 0 \quad (n=1,2,\ldots,n'_n), \qquad \sum_{k=1}^{n_e} [k,m]\, V_k = 0 \quad (m=1,2,\ldots,n_m), \qquad (2)$$

where, of course, $\Delta Z_{kl} = Z'_{kl} - Z_{kl}$ *will be zero when k and l are different from h; and* $\Delta I_k = I'_k - I_k$ *will be zero when there is a current source in the element k. Putting (also):* $\Delta V_k = V'_k - V_k$, $\Delta D_k = D'_k - D_k$, *we will have (subtracting eqs. 1 from eqs. 2):*

$$V_k = \sum_{l=1}^{n_e} Z'_{kl}\cdot\Delta I_l + \sum_{l=1}^{n_e} I_l\cdot\Delta Z_{kl} - \Delta D_k \qquad (k=1,2,\ldots,n_e),$$
$$\sum_{k=1}^{n_e} (k,n)\Delta I_k = 0 \quad (n=1,2,\ldots,n'_n), \qquad \sum_{k=1}^{n_e} [k,m]\Delta V_k = 0 \quad (m=1,2,\ldots,n_m). \qquad (3)$$

But these equations are the complex equations of the network with the modified impedances, with the increments in the complex currents and voltages (=effects of the impedance changes) as the complex currents and voltages, with all the current sources nullified, and with the complex values of the voltage sources replaced by the following compensating electromotive forces:

$$E'_k = -\sum_{l=1}^{n_e} \Delta Z_{kl}\cdot I_l \qquad (\text{i.e., } E'_h = -\sum_{l=1}^{n_e} \Delta Z_{hl}\cdot I_l \ \&\ E'_k = -\Delta Z_{kh}\cdot I_h,\ k\neq h), \qquad (4)$$

where the I_l *are the complex currents in the original network (before the modification). Therefore, the effects of the changes in the self- and mutual-impedances of the element h are the same as the effects of the compensating voltage sources of complex electromotive forces* E'_k *(given by eqs. 4) inserted in (series with) the elements* $k(=1,2,\ldots,n_e)$ *of the network with the modified impedances (and with the sources of the original network nullified).*

In particular, when only the self-impedance (of the box) in the element h is changed (by an amount ΔZ*), so that* $\Delta Z_{kl} = 0$ *if* $k \neq h$ *or* $l \neq h$*, and* $\Delta Z_{hh} = \Delta Z_h = \Delta Z$*, we will have:* $E'_k = 0$ *if* $k \neq h$*, and* $E'_h = -\Delta Z\cdot I_h$*; and so, in this case, the effects due to a change* ΔZ *in the self-impedance of the (box in) element h are the same as the effects of a single voltage source of complex value* $-\Delta Z\cdot I_h$ *inserted in (series with) the element h, where* I_h *is the complex current in the branch h of the original (un-modified) network; and this is precisely the usual form of the compensation theorem.*

Corollary: A part ΔZ of a magnetically isolated impedance Z through which the complex current is I can be replaced by a voltage source with a complex EMF. $E = -I\cdot\Delta Z$ (i.e., with the same voltage drop as ΔZ) in the reference direction for I. Because $Z = (Z - \Delta Z) + \Delta Z$.

Problem. Show that in an arbitrary non-singular sinusoidal current network, considered as a network of passive box-source parallel elements (with the complex equations (1 & 2) of Ch. VIII, §4), if the self- and mutual-admittances of the box of the element h (only) are changed by amounts ΔY_{hl}, and the network remains non-singular and resumes the sinusoidal state, the effects of these changes are the same as the effects of compensating current sources of complex currents:

$$I'_{cs_h} = \sum_{l=1}^{n_e} \Delta Y_{hl}\cdot V_l\,, \qquad I'_{cs_k} = \Delta Y_{kh}\cdot V_h \quad (\text{for } k \neq h),$$

shunted across the elements k$(=1,2,\ldots,n_e)$, where the V_l $(l=1,\ldots,n_e)$ are the complex voltage drops across the parallel elements in the original (unmodified) network. In particular, when only the self-admittance of the element h is changed, we will have: $I'_{cs_k}=0$ if $k\neq h$, and $I'_{cs_h} = \Delta Y_{hh}\cdot V_h$; and the effects of the change are the same as the effects of a single compensating current source of complex current $I'_{cs_h} = \Delta Y_{hh}\cdot V_h$, shunted across the element h (as modified). (Of course, to compute the effects of the compensating current sources, it is best to nullify all the sources in the given network, make the modifications of the admittances of the element h, and then shunt the compensating current sources across the admittances as modified. The complex currents and voltages in this, so constructed, network will be the effects sought, according to the superposition principle.)

§ 3. THE INSERTION OF SOURCES.

Theorem 1. If a sinusoidal voltage source is connected between two mutually accessible points a and b of a given sinusoidal current network of the same frequency (which is and remains non-singular after the insertion) between which there is a complex voltage drop V_{ab}, and if the complex electromotive force E (say) of the voltage source, from a to b, is equal to the complex voltage rise $-V_{ab} = V_{ba}$ from a to b, then the effects of the insertion of the voltage source are null. That is, the complex currents and voltages in the network remain unaltered by the insertion, and the complex current I (say) through the inserted voltage source shall be zero.

Proof: This is done by the guess and check method: We guess that the effects of the insertion of the voltage source shall be null, and then we check that this is true by substitution in the equations of the network; finally, we make use of the existence *and uniqueness of the currents and voltages in a non-singular network to show that the guess is possible and that it is the only one.*

Let us assume that we have the set of all the complex currents and voltages in the given network, which exists (not meaning that the currents and voltages are $\neq 0$) because it is non-singular. The complex currents of this set, together with $I=0$, will clearly satisfy all of Kirchhoff's complex current laws for the nodes of the new network (augmented by the inserted voltage source), because they remain essentially the same as for the old (given) network. Moreover, the complex voltages of the set, will clearly satisfy all of Kirchhoff's complex voltage laws for the meshes in the new network that can also be found in the old one, because they remain the same; while for the new meshes, they are guaranteed because they can be obtained from the previous ones with the help of the equation: $-E+V_{ba}=V_{ab}+V_{ba}=0$, which is true by hypothesis. Furthermore, all the equations expressing the complex voltage drops through the elements in the new network which can be found in the old one, hold also; because they are the same as in the old network; while for the new element (the inserted voltage source) this is true because $V_{ab}=-E$, by hypothesis. Thus the check and the proof are completed.

<u>Theorem 2</u>. If a sinusoidal current source is inserted in (series in) a branch $\underline{B}$, of a given sinusoidal current network of the same frequency (which is and remains non-singular after the insertion), which carries a complex current I_B (say), and if the complex current $\underline{I}$ (say) through the current source is equal to I_B, then the effects of the insertion are null. That is, the complex currents and voltages in the network shall remain unaltered by the insertion of the current source, and the complex voltage rise $\underline{D}$ (say) through the inserted current source shall be zero.

<u>Proof</u>: This is also done by the guess and check method. Let us assume that we have the set of all the complex currents and voltages in the given network, which exists because it is non-singular. The complex currents of this set will clearly satisfy Kirchhoff's complex current laws for all the nodes in the new (augmented) network which were in the old (given) one, because they are the same as for the old network; while for the new nodes (=the terminals of the inserted current source) they are guaranteed because $-I+I_B=0$ by hypothesis. Moreover, the complex voltage drops of the set, together with $D=0$, will clearly satisfy all of Kirchhoff's

complex voltage laws for the meshes of the new network, because they are essentially the same as (they differ only in the addition of D=0 to some of them) in the old network. Furthermore, all the equations expressing the complex voltage drops through the elements in the new network hold also, because they remain essentially the same (except for a possible addition of D=0 in one of them), as for the old network. This completes the check. Finally, they are the only possible complex currents and voltages in the new network, because it is non-singular by hypothesis. This completes the proof.

Theorem 3. The effects of the insertion of a sinusoidal voltage source (in series) in a branch or a current source (in parallel) across a branch, in any given network in the sinusoidal state of the same frequency (assumed to remain non-singular) are the negatives of the effects of the insertion of the same source (but) in the opposite sense (i.e., with the terminals inverted).

Proof: In order to find the effects of the inserted source, according to the superposition principle, we nullify all the sources originally in the given network, and then we insert the source in question and find the complex currents and voltages it produces in this network. These are the effects of the inserted source. Now consider the complex equations of this network (which contains a single source) with the source inserted in one sense. Changing the signs of all the terms in these equations will yield the complex equations of the network with the source inserted in the opposite sense, as can easily be checked. Therefore, the negatives of the complex currents and voltages in the former network satisfy the complex equations of the latter; and because of the uniqueness of the complex currents and voltages in a non-singular network they are the only possible ones. This shows that the effects in the former network (with the source inserted in one sense) are the negatives of the effects in the latter network (with the source inserted in the opposite sense), which completes the proof.

Theorem 4. If in a given non-singular sinusoidal current network, two equal and oppositely series connected sinusoidal voltage sources of the same frequency (as shown in Fig. 1) are inserted in a branch, the effects of the insertion are null.

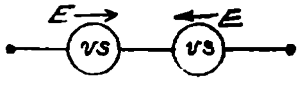


Fig. 1.

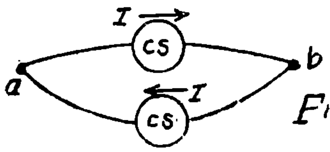


Fig. 2.

Proof: It is easy to see that the equations of the network remain (essentially) unaltered by the insertion of the voltage sources. This means that the complex currents and voltages in the old (given) network satisfy the equations of the new network (augmented by the inserted voltage sources). And since the network is (assumed to be) non-singular, these complex currents and voltages are then the only possible ones in the new network. This in turn means that the complex currents and voltages remain unaltered by the insertion of the sources, which means that the effects of the insertion are null.

Theorem 5. If in a given non-singular sinusoidal current network, two equal but oppositely parallel connected sinusoidal current sources (as shown in Fig. 2) of the same frequency are connected between two points a & b of the network, the effects of the insertion are null.

Proof: If we write the equations of the old (given) network and those of the new network (augmented by the inserted current sources), we see that they are essentially the same. This means that the complex currents and voltages in the old network satisfy the equations of the new network; but since the network is non-singular, these complex currents and voltages are the only possible ones in the new network. This means that the complex currents and voltages remain unaltered by the insertion of the sources, and this means that the effects of the insertion are null.

Problem. Show that if all the sources in a non-singular network are inserted in the opposite directions then all currents & voltages are changed in sign (only).

§4. THE THEOREM OF HELMHOLTZ AND THEVENIN.

In Ch. VIII, §5, we have already given an account of this theorem and of its generalization. Nevertheless, here we will again consider the (simple) form of the theorem from another interesting point of view. For this purpose, consider a non-singular network in the sinusoidal state, consisting of an active two-terminal structure (box A) with a finite open-circuit passified impedance Z_A (= impedance of the structure A with all its sources nullified), connected to an arbitrary passive two-terminal structure (box B) with mutually accessible terminals a & b and impedance Z_B, as shown in Fig. 1, assuming that these structures are not coupled magnetically.

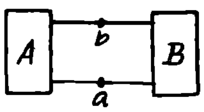


Fig.1.

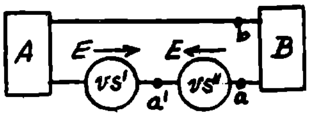


Fig.2.

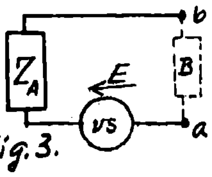


Fig.3.

Let E denote the open-sircuit complex voltage rise from a to b of the box A (with the box B removed). By theorem 4, §3, we can insert two equal and oppositely connected voltage sources vs' & vs" of complex electromotive forces E, as shown in Fig. 2, without altering the currents and voltages. Now let us apply the superposition principle to the augmented network (of Fig. 2), with all the sources of the network grouped into two classes: (1º) that of all the sources within the active box A together with the voltage source vs', and (2º) that of the single source vs". By the superposition principle, the complex currents and voltages in the given network shall be the sums of the corresponding quantities of the two auxiliary networks obtained by nullifying the sources of one class at a time. When vs" is nullified (shorted) and the network is left with the sources of the first class, we will first prove (by the guess and check method) that the complex currents and voltages in, and through, the box B are null. But this is easy, because the hypotheses that all the currents and voltages in, and through, the passive box B are null, and that the part of the network (of Fig. 2) to the left of a' & b is just the same as if it were opened at these points, are clearly consistent with the equations of the network (since E was chosen to be the open circuit complex voltage of the structure A); and by the existence and uniqueness theorems for non-singular networks, they are the only possible ones. As a consequence, all the currents and voltages in the passive box B shall be due exclusively to the single voltage source vs", when all the rest of the sources in the network (of Fig. 2), of the first class, are nullified. But this means that all the currents and voltages in the arbitrary passive box B will be the same as when it is connected to a voltage source of complex electromotive force E in series with the passified box of impedance Z_A, as shown in Fig. 3. And since the passive box B is arbitrary, this means that the active box A can be replaced by the series connection of a single voltage source of complex electromotive force E (=open-circuit complex voltage of the active box A) and a passive box of impedance Z_A (=impedance of box A when all its sources are nullified); and in particular, the complex current flowing into box B will then be: $E/(Z_A+Z_B)$. And this is precisely the ordinary form of the theorem of Helmholtz and Thévenin.

§5. THE THEOREM OF HELMHOLTZ AND NORTON.

In Ch. VIII, §5, we have already given an account of this theorem and of its generalization; but now we wish to give another interesting proof of its simple form. For this purpose, consider a non-singular network in the sinusoidal state, consisting of an active two-terminal structure (box A) with a finite short-circuit passified admittance Y_A (=admittance of the structure A when all its sources are nullified), connected to an arbitrary passive two-terminal structure (box B) with mutually accessible terminals a & b and admittance Y_B, as shown in Fig. 1, assuming that these structures are not coupled magnetically.

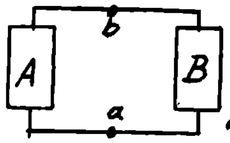


Fig.1.

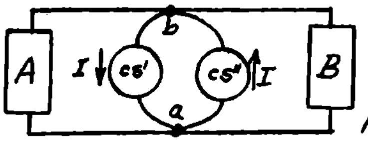


Fig.2.

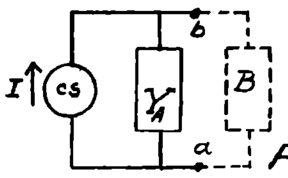


Fig.3.

Let I denote the short-circuit complex current through the box A from a to b (with the box B removed and a & b connected together). By theorem 5, §3, we can insert two equal but oppositely connected current sources cs' and cs" between a & b, of complex currents I, as shown in Fig. 2, without altering the currents and voltages in the given network. Now let us group all the sources in the augmented network (of Fig. 2) into two classes: (1º) that of all the sources within the active box A together with the current source cs', and (2º) that of the single source cs". By the superposition principle, we know that the complex currents and voltages in the (augmented and hence also in the given) network shall be the sums of the corresponding quantities of the networks obtained by nullifying the sources of one class at a time. When cs" is nullified (opened) and the network is left with the sources of the first class, it is easy to prove (by the guess and check method) that all the complex currents and voltages in and through the box B are null; because the hypotheses that all the currents and voltages in the box B are null, and that the part of the network (of Fig. 2) to the left of a & b is just the same as if it were shorted at these points, are clearly consistent with the equations of the network (since I was chosen to be the short-circuit complex current of the box A); and by the existence and uniqueness theorems for non-singular networks, they are the only possible ones. Thus all the currents and voltages in the passive box B will be due only to the single current source cs" acting alone, with all the rest of the sources in the augmented network (of Fig. 2), of the first class, nullified. But this means that all the currents and voltages in the arbitrary passive box B will be the same as when it is connected across a current source of complex current I in parallel with the admittance Y_A of the passified box A, as shown in Fig. 3. And since the passive box B is arbitrary, this means that the active box A can be replaced by the parallel connection of a single current source of complex current I (=short-circuit complex terminal current of the active box A) and a passive box (box A with all its sources nullified) of admittance Y_A; and in particular, the complex voltage drop across the box B will clearly then be: $I/(Y_A+Y_B)$. But this is precisely (the simple form of) the theorem of Helmholtz and Norton, ~~qed~~.

§ 6. THE SHORT-CIRCUIT THEOREM.

When two mutually accessible points (taken as nodes on the same component) of a network are joined together directly (i.e., identified, or combined into a single node, or, as it is sometimes descriptively put, connected together by an "impedanceless connection"), we say that they have been short-circuited, or simply shorted. Usually the definition is restricted to cases in which the two points are at distinct potentials; but this restriction is clumsy and unnecessary. The two points must not, however, be the terminals of an ideal voltage source of non-vanishing value; for otherwise an inconsistency would arise. The short-circuit may be produced intentionally (as when a switch is closed), or accidentally; and it may or may not be of disastrous consequences (all this being irrelevant to the theory). By a multiple (or compound) short-circuit we mean the simultaneous identification (=direct connection) of more than two mutually accessible points of a network.

The short-circuit theorem (which seems to have been mentioned first, without a proof, by J. R. Carson in his book on Electric Circuit Theory and Operational Calculus (p.160), of 1926) states that the effects (=increments in currents and voltages) due to a (simple) short-circuit between the points a & b of a given non-singular network (which remains non-singular) are the same as the effects of a single voltage source connected between a & b with a voltage rise from a to b equal (at all subsequent time) to the voltage drop from a to b that would exist in the absence of the short-circuit.

That is, the effects of the short-circuit between a & b are the same as the currents and voltages produced in the network, when all its sources are nullified, by a single voltage source connected between a & b which produces the negative of the voltage that would otherwise exist between them had the short-circuit not been made.

Of course, here we are dealing with networks in the sinusoidal state at a given frequency, only, and so the voltage between a & b will be sinusoidal in the absence of the short-circuit; and hence the voltage source producing the same effects as the short-circuit will also be sinusoidal, of the same given frequency. This sinusoidal voltage source shall, however, produce certain transients which need not be sinusoidal, and which may sometimes be the most important parts of the effects of the short-circuit. Nevertheless, these transients are not treated here; and we imagine that they have died out (without destroying the network) and that the sinusoidal state has been resumed.

Our task is then to prove that the increments in the complex currents and voltages in the non-singular network (=effects) due to the short-circuit between a & b are the same as the effects due to a single voltage source connected between a & b of complex electro-motive force E (from a to b) equal to the complex voltage drop V_{ab} (from a to b) that would exist if the short-circuit had not been made. These effects may then be found by nullifying all the sources in the given network, and then finding the complex currents and voltages produced by the single voltage source E connected from a to b. These are the increments produced by the short-circuit, and so if they are added to the corresponding complex currents and voltages that were in the network before the short-circuit, we will have the complex currents and voltages in the network after the short-circuit is made.

To prove this let us insert between a & b a voltage source of complex elctromotive force equal to the complex voltage rise between these points (sense and all). By theorem 1, §3, we know that this will have no effects on the complex currents and voltages in the given network. Now let us group the sources in the new (augmented) network so obtained into two classes: (1º) that of all the sources of the given network, and (2º) that of the new (inserted) voltage source only. By the superposition principle, the complex currents and voltages in the augmented (and therefore in the given) network will be the sums of the corresponding quantities of the two auxiliary networks obtained by nullifying the sources of one class at a time. Therefore, the complex currents and voltages in the given network are equal to the corresponding quantities in the network with the short-circuit (which is the network obtained from the augmented network by nullifying the newly inserted voltage source) added to the corresponding quantities in the new network when all its sources (of the first class) are nullified except the newly inserted voltage source. That is, the original complex currents and voltages in the given network minus the corresponding quantities in the network with the short-circuit (which are precisely the negatives of the effects of the short-circuit) are equal to the effects of the newly inserted voltage source, when all the sources in the given network are nullified. Consequently, by reversing the connections of the new voltage source (i.e., inserting it in the opposite sense) it will produce the same effects in the given network as the short-circuit (see the problem at the end of §3) ; which is precisely the statement of the short-circuit theorem (for sinusoidal current networks).

§7. THE OPEN-CIRCUIT THEOREM ON THE EFFECTS OF AN INTERRUPTION.

Roughly, an interruption or break is the opposite of a short-circuit. It may be defined, without loss of generality, as the splitting of a node of a network into two nodes. The break must not be, however, in a branch with a current source of non-zero current; for otherwise an inconsistency would arise. An interruption may be produced intentionally (as when a switch is opened), or accidentally (as when there is a break in a conductor); but this is irrelevant to the theory. A generic name for short-circuits and interruptions is a fault (even ~~if~~ though one is inclined to consider a fault as distructive, or at least accidental). By a multiple (or compound) interruption or break we mean the simultaneous splitting of a node (or of several nodes) into several nodes.

We will now prove the following theorem on an interruption, or opening of a switch, for sinusoidal current networks (due in essence to J. R. Carson, it seems --- see note in §6).

The effects (=increments in the complex currents and voltages) due to an interruption (=break) at some place (e.g., in a branch) of a given non-singular sinusoidal current network (which remains so), which was to carry (in the absence of the break) a complex current I, are the same as the effects of a current source *of complex current I inserted in the broken place, in the opposite sense to which I was referred. The effects of the interrruption will then be the complex currents and voltages in the network obtained from the given network by nullifying all its sources and then inserting a current source, in the place of the break, of a complex current equal but opposite to that which would otherwise exist in the absence of the break.*

To prove this, let us insert a current source, in the place where the break is to be made, of a complex current equal to that which would exist in this place in the absence of the break. By theorem 2, §3, we know that this will have no effects on the complex currents and voltages in the given network. Now let us group the sources in this new (augmented) network, so obtained, into two classes: (1º) that of all the sources of the original (given) network, and (2º) that of the newly inserted current source alone. By the superposition principle, the complex currents and voltages in the augmented (and therefore also in the given) network will be the corresponding sums of the effects of the sources of these two classes. But the network with the sources of the first class acting, and with the single (current) source of the second class nullified, is precisely the network with the break (which we assume to have regained the sinusoidal state). Therefore, the complex currents and voltages in the given network are equal to the corresponding quantities in the network with the break, added to those in the network obtained by nullifying all the sources in the given network (those of the first class) and inserting the current source (of the second class) in the break. That is, the original complex currents and voltages in the given network minus the corresponding quantities in the network with the break (which are precisely the negatives of the effects of the break) are equal to the currents & voltages produced by the newly inserted current source (that of the second class), when all the sources (of the first class) in the given network are nullified. Consequently, by reversing the connections of the new current source (i.e., inserting it in the opposite sense) it will produce the same effects in the given network as the break, according to the problem at the end of §3; and this is precisely the statement of the open-circuit theorem (for sinusoidal current networks), the proof of which is thus complete.

§8. THE EXCHANGE OR TRANSFORMATION OF SOURCES.

Consider a voltage source in series with a magnetically isolated passive box of impedance $Z \neq 0$ (and admittance $Y=1/Z$), as shown in Fig. 1(a), to be part of a network in the sinusoidal state. The complex voltage drop from the terminal a to the terminal b is:

$$V_{ab} = ZI_{ab} - E, \qquad (1)$$

where I_{ab} is the complex current entering a and leaving through b, and E is the complex electromotive force of the voltage source.

On the other hand, consider exactly the same box but now in parallel (shunted) with a current source of complex current I, as shown in Fig. 1(b), to be **substituted** *for the structure of Fig. 1(a) in the given network. The complex voltage drop from the terminal a to the terminal b (which is now* **equal**, *in this case, to the voltage drop through the box carrying the complex current $I'=I_{ab}-I$) is:*

$$V_{ab} = ZI' = ZI_{ab} - ZI. \qquad (2)$$

Equations (1) and (2) will be identical if and only if:

$$E = ZI, \text{ or } I=YE. \qquad (3)$$

That is, the two structures of Fig. 1(a & b) will have identical terminal characteristics (at the frequency of the given network) if and only if the voltage source in Fig. 1(a) has a complex electromotive force E, and the current source in Fig. 1(b) has a complex current I, which satisfy the equation (3); and in this case, equal complex voltage drops will imply equal complex terminal currents through the two structures, and viceversa; and so equal complex voltage drops will be equivalent to equal complex terminal currents through the structures, so that one may be exchanged with the other without altering the equations of the given network, and hence without altering the complex currents and voltages in the other elements of the network (if it is non-singular, of course). This result is the well-known theorem on the exchange, or transformation, of sources, which finds many practical applications.

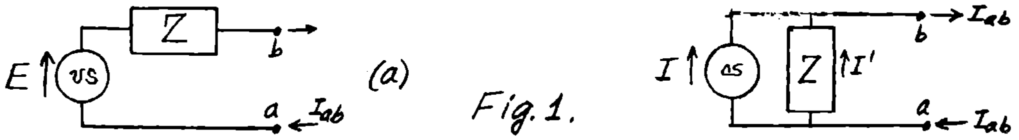


Fig. 1.

Problem 1. With the help of eqs. (6) of Ch. VIII, §5, show that the above result holds even for active boxes with non-vanishing finite passified impedances.

Problem 2. With the help of the source exchange theorem, obtain the (simple) theorem of Helmholtz and Norton (given in §5) from the (simple) theorem of Helmholtz and Thévenin (given in § 4), and the converse also.

Problem 3. Show how the networks shown in Fig. 2(a & b) can easily be solved (i.e., how the complex currents and voltages can be found) by making source exchanges.

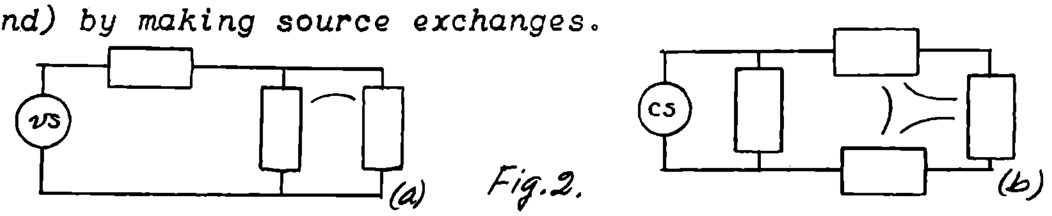


Fig. 2.

The above result on the exchange of sources can easily be generalized to the case in which the boxes in series with the voltage sources are active and magnetically coupled. For this purpose, let us consider a sinusoidal current network with N structures as shown in Fig. 3, consisting each in a voltage source in series with an active or passive box, and let us assume that the N boxes of these structures are magnetically coupled with each other (but with no other boxes, thus forming a magnetically closed system of boxes).

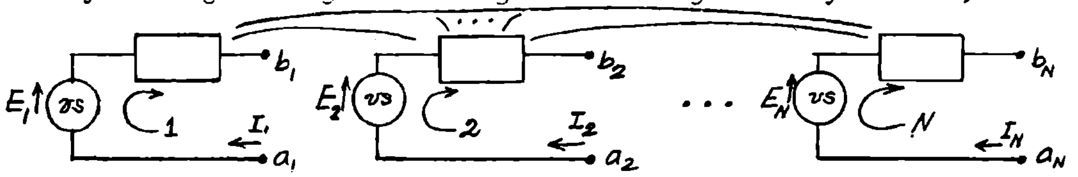


Fig. 3.

The complex voltage drop from the terminal a_k to the terminal b_k of the typical box k (=1,2,...,N) is, according to eqs. 3, Ch. VIII, § 5:

$$V_k = \sum_{l=1}^{N} Z_{kl} I_l + V_k^\circ - E_k \qquad (k = 1,2,\ldots,N), \tag{4}$$

where the Z_{kl} are the (self- and mutual-) passified impedances of the system of boxes (with all their internal sources nullified), I_k is the complex terminal current entering through a_k and leaving through b_k, V_k° is the open-circuit complex voltage drop through the box k, and E_k is the complex electromotive force through the "explicit" voltage source of the structure k; everything being referred to the assigned orientations (=reference directions).

On the other hand, consider exactly the same N active boxes of the system, with the same magnetic couplings, but now in parallel (shunted) with current sources of complex currents I_{cs_k} (k=1,2,...,N), as shown in Fig. 4. The complex voltage drops will be the same as those through the active boxes, namely (according to eqs. 3, Ch. VIII, §5):

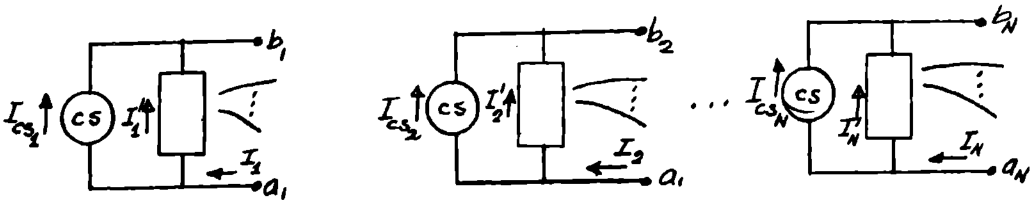


Fig. 4.

$$V_k=\sum_{l=1}^{N} Z_{kl}I'_l+V^\circ_k=\sum_{l=1}^{N} Z_{kl}(I_l-I_{cs_l})+V^\circ_k=\sum_{l=1}^{N} Z_{kl}I_l-\sum_{l=1}^{N} Z_{kl}I_{cs_l}+V^\circ_k \quad (k=1,2,\cdots,N), \cdots\cdots(5)$$

since now the complex currents through the boxes are: $I'_k=I_k-I_{cs_k}$, *where the* I_k *are again the complex terminal currents through the structures* $(k=1,2,\cdots,N)$.

Equations (4) and (5) will be identical if and only if:

$$E_k=\sum_{l=1}^{N} Z_{kl}I_{cs_l} \qquad (k=1,2,\ldots,N). \qquad (6)$$

That is, the two systems of structures of Figs. 3 & 4 will have identical terminal characteristics (at the frequency of the given network) if and only if the complex electromotive forces E_k *of the voltage sources in Fig. 3, and the complex currents* I_{cs_k} *through the current sources in Fig. 4, satisfy the equations (6); and in this case, equal complex voltage drops will imply equal complex terminal currents through the two systems of structures, and viceversa; thus equal complex voltage drops will be equivalent to equal complex terminal currents through the two systems of structures, and so one system may be exchanged with the other without altering the equations of the given network, and hence without altering the complex currents and voltages through the other structures of the network (assuming that it is non-singular, of course). This result is a substantial generaliztion of the (simple) theorem on the exchange, or transformation, of sources given above, which is indeed very useful.*

When $\det(Z_{kl})\neq 0$, *eqs. (6) can be written in the following equivalent form:*

$$I_{cs_k}=\sum_{l=1}^{N} Y_{kl}E_l \qquad (k=1,2,\ldots,N), \qquad (7)$$

where the $Y_{kl}=\operatorname{cof} Z_{lk}/\det(Z_{kl})$ *are the (self- and mutual-) admitances of the passified boxes of the system (with their sources nullified).*

Problem 4. Show that a sinusoidal current network of structures of the kind shown in Fig. 5 (or of the kind shown in Fig. 6) can always be transformed into a network of box-source series element of the kind shown in Fig. 7, and also into a network of box-source shunt elements of the kind shown in Fig. 8.

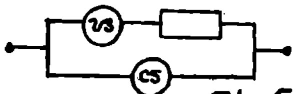


Fig. 5.

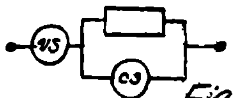


Fig. 6.

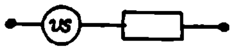


Fig. 7.

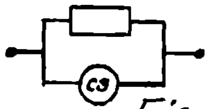


Fig. 8.

CHAPTER X: THE DUALITY PRINCIPLE.

By this time the careful reader may have noticed a certain relation between the treatment (and equations) of an arbitrary network of series elements of one kind or another (that we have considered in the preceding chapters) and the treatment (and equations) of an arbitrary network of parallel (or shunt) elements of the corresponding kind. Thus, making abstraction of the detailed structures of the networks, it may be checked that by making the interchanges of corresponding quantities (in juxtaposition) of the following table (using the notation of the text):

series	shunt	series	shunt	series	shunt	series	shunt
V_k	I_k	R_k	G_k	(k,n)	$[k,m]$	$\frac{d()}{dt}$	$\int()dt$
I_k	V_k	S_k	C_k	$[k,m]$	(k,n)	$\int()dt$	$\frac{d()}{dt}$
E_k	$-I_{cs_k}$	L_{kl}	Γ_{kl}	n'_n	n_m	$i\omega$	$1/i\omega$
D_k	$-I_{vs_k}$	Z_{kl}	Y_{kl}	n_m	n'_n	$1/i\omega$	$i\omega$
J_m	U_n	X_{kl}	B_{kl}	n_{cs}	n_{vs}	$p(=\sigma+i\omega)$	$1/p$
$\tilde{E}_m$	$\tilde{I}_n$	z_{mn}	y_{mn}			$1/p$	p
E°_k	I°_k						

the generic equations and solution formulae of an arbitrary network of series elements (e.g., eqs. 3, §1, Ch. V, and eqs. 1,2,3,4, §1, & 1,3,4, §2, & 14,15, §4, of Ch. VI) are transformed into the generic equations and solution formulae of an arbitrary network of parallel (or shunt) elements (e.g., eqs. 3, §2, Ch. V, and eqs. 0,1, 2,3, §1, & 1,2,3, §2, & 5,6, §4, of Ch. VII), and viceversa.

This statement may properly be called the abstract duality principle, since it is based on a correspondence between symbols pertaining to generic (in contrast to specific) elements and networks. And corresponding quantities in the above table are called dual quantities, or simply duals.

Of course, the proof of this statement is in great part the contents of the preceding chapters; so that this result could hardly have been used to reduce the material presented. And once the material has been presented it is just as easy to write down the general equations of one network as those of the other. Thus this result turns out to be too abstract to be of much use, unless it is used as a memory guide. Naturally this was to be expected of such a general result, since it was based on representative elements and overall characteristics of the networks, all individual characteristics being lost by the abstraction. As a consequence it is to be expected that we must make some kind of restrictions in order to continue; and this will be done in the rest of this chapter.

§1. THE DUALITY THEOREM.

The general complex equations of an arbitrary network of n_e general series elements interconnected into n_c components (=separate parts) with n_n nodes (and $n_n'=n_n-n_c$ independent nodes) and $n_m=n_e-n_n'$ independent meshes are the following (cf. eqs. 3, §1, Ch. V):

$$\begin{aligned} &V_k = R_k I_k + S_k I_k/i\omega + \sum_{l=1}^{n_e} i\omega L_{kl} I_l - E_k - D_k \qquad (k=1,2,\ldots,n_e), \\ &\sum_{k=1}^{n_e}(k,n)I_k = 0 \qquad (n = 1,2,\ldots,n_n'), \qquad\qquad (1) \\ &\sum_{k=1}^{n_e}[k,m]\,V_k = 0 \qquad (m = 1,2,\ldots,n_m); \end{aligned}$$

and the mesh method is introduced by making the substitution:

$$I_k = \sum_{m=1}^{n_m}[k,m]\,J_m \qquad (k = 1,2,\ldots,n_e). \qquad (2)$$

Similarly, the general complex equations of an arbitrary network of N_e general parallel elements, interconnected into N_c components with N_n nodes (and $N_n'=N_n-N_c$ independent nodes) and $N_m=N_e-N_n'$ independent meshes are the following (cf. eqs. 3, §2, Ch. V):

$$\begin{aligned} &I_k = G_k V_k + i\omega C_k V_k + \sum_{l=1}^{N_e}\Gamma_{kl}V_l/i\omega + I_{cs_k} + I_{vs_k} \qquad (k=1,2,\ldots,N_e), \\ &\sum_{k=1}^{N_e}(k,n)^* I_k = 0 \qquad (n = 1,2,\ldots,N_n'), \qquad\qquad (3) \\ &\sum_{k=1}^{N_e}[k,m]^* V_k = 0 \qquad (m = 1,2,\ldots,N_m); \end{aligned}$$

where $(k,n)^*$ and $[k,m]^*$ denote the incidence numbers of the elements with the (independent) nodes and meshes, respectively, of this network; the node method being introduced by the substitution:

$$V_k = \sum_{n=1}^{N_n'}(k,n)^* U_n \qquad (k = 1,2,\ldots,N_e). \qquad (4)$$

If $n_e=N_e$, the number of general series elements of the former network will be equal to the number of general parallel elements of the latter, and a one to one correspondence can be established between the elements of the two networks. Further, if $n_n'=N_m$ and $N_n'=n_m$, then the number of independent nodes of one network will be equal to the number of independent meshes of the other, and a one to one correspondence can also be established between the independent nodes of each network with the independent meshes of the other.

Under these conditions, the number of (independent) equations of the first network will be the same as that of the second network; and if (and only if) for some enumeration of the elements, nodes, and meshes, and for some orientation of the elements and meshes, of the two networks we have: $(k,n) = [k,m]^*$ and $[k,m] = (k,n)^*$ for all the corresponding elements, nodes, and meshes, then the systems of equations (1) & (3) of the two networks become interchanged upon interchanging (all) the corresponding (dual) quantities according to the table given above; the columns on the left sides referring to the (first) network of general series elements, and the columns on the right sides referring to the (second) network of general parallel (shunt) elements. Moreover, this interchange of equations of one network with those of the other is effected in a one to one manner; with the equation expressing the voltage drop across an element in one of the networks passing over into the equation expressing the current through the corresponding element of the other network, and the equation expressing Kirchhoff's current law for a node of one network passing over into the equation expressing Kirchhoff's voltage law for the corresponding mesh of the other network, as can easily be checked (under the assumed conditions) by a glance at eqs. (1) and (3). This result is called the duality theorem; and the two networks, one considered as a network of general series elements and the other considered as a network of general parallel (shunt) elements, are called dual networks.

Of course, since any network may be considered as a network of general series elements, and also as a network of general parallel elements, it makes no difference which is considered as one and which is considered as the other, as long as the conditions of the theorem are met with. Moreover, it is not necessary that corresponding quantities (duals) of the two networks be the same numerically (of course); rather it should be understood that their interchange, as indeterminates (or symbols), will interchange the corresponding network equations in a one to one way, in the manner mentioned above. When the equations for one of a pair of networks known to be duals are known, the equations for the other network (for some enumeration and orientation of its elements, nodes, and meshes) can be obtained immediately by substituting all the dual quantities in the given equations, in accordance with the table given above. Notice that according to this table, voltage and current sources are interchanged, but passive elements are changed into the same kind of elements. (Later we will consider another (more restricted) kind of duality, which is the one commonly found in the literature, in which passive elements of one kind are changed into passive elements of another kind.) The principle involved in all this is called the (general) duality principle.

When two quantities A & B are defined for two dual networks as functions of corresponding given dual quantities, and these functions become interchanged when all the given dual quantities are interchanged, then the quantities A & B are also called dual quantities, or simply duals.

Thus, from eqs. (15), §1, and eqs. (5), §2, of Ch. V, we see that the impedances Z_{kl} of (and between) the elements of the network of general series elements, and the admittances Y_{kl} of (and between) the elements of the dual network of general parallel elements, are dual quantities. This was anticipated when the table of dual quantities given in the introduction to this chapter was established, of course, and for this reason this pair was included in the table. Likewise, according to eqs. (1), §2, Ch. VI, and eqs. (1), §2, Ch. VII, the same can be said of the mesh impedances z_{mn} and the nodal admittances y_{mn} of a pair of dual networks, and for this reason this pair was also included in the table. And the same can be said about the (complex) mesh or circulating currents J_m and nodal potentials U_n, introduced by eqs. (2) & (4); and also about the other quantities (of the mesh and nodal methods, etc.) which were included in the table.

§2. THE CONSTRUCTION OF DUAL NETWORKS.

The equations: $[k,m]=(k,n)^*$ and $[k,m]^*=(k,n)$, between the element-mesh and element-node incidence numbers of a pair of dual networks, imply that it is possible to choose the independent meshes of each of these networks in such a way that no element belongs to more than two independent meshes; because, for a given element k, (k,n) and $(k,n)^*$ are non-zero for at most two distinct nodes (values of) n; and so the same must be true of $[k,m]$ and $[k,m]^*$ which are equal to them. Now it can be shown that this implies that the networks must be planar (cf. H. Whitney, Planar graphs, Fundamenta Mathematicae (21), (Warsaw) 1933); that is, that their graphs may be mapped upon a plane without elements crossing. And since the converse of this implication is clearly true, we see that the duality theorem of §1 is limited to networks (the graphs of) which are planar.

Let us now consider a given planar network drawn upon a plane without elements crossing, and let us take the "openings" as the meshes of the network. Also, let us take the base node in each component of the network (re-numbering the nodes if necessary) to be on the outside boundary of the corresponding component, as shown in Fig. 1. In each opening (mesh) in the given network let us take a point as a prospective node for a new network, and for each component in the given network let us take a point outside of the network as a prospective base node for a corresponding component in the new network. In this way we will be sure that the number of independent nodes of the new network will be equal to the number of independent meshes of the given network; and we will now show how the prospective nodes of the new network can be joined together in such a way that the new network will be planar and have the same number of elements, the same number of components, and a number of independent meshes equal to the number of independent nodes of the given network.

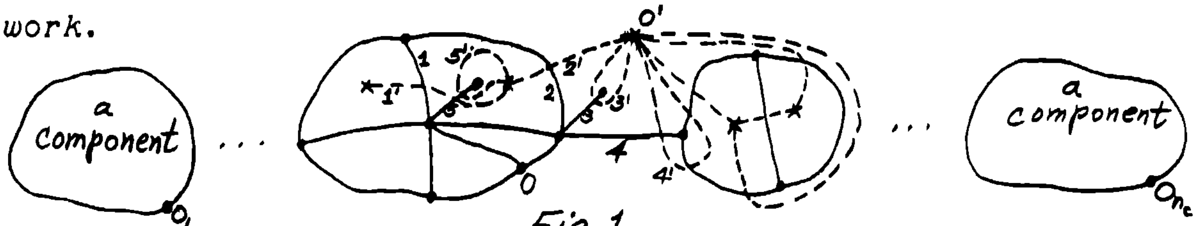


Fig. 1.

In the first place, for each element of the given network which is common to two of its openings (meshes), let us construct an element of the new network, joining the prospective nodes enclosed by these openings, across their common (boundary) element. Next, for each element on the boundary of a given component and belonging to a single mesh of the given network, let us construct an element of the new network crossing the boundary element and joining the prospective node enclosed by this single mesh to the prospective base node corresponding to the given component (taking care, as the construction progresses, not to enclose any of the base nodes of the given network in a mesh of the new network, which is always possible). Finally, for each non-essential element with a loose terminal (belonging to no mesh) on a given component of the given network, let us construct an element of the new network starting from the corresponding prospective base node, or from the prospective node of the new network within the same opening of the given network in which the loose terminal lies, then crossing the non-essential element, and then returning (back) to the starting prospective node of the new network; and for the only non-essential element with no loose ter-

minal which may exist on a given component of the given network (whose removal would split this component in two; one part P of which would not contain the base node of the given component), let us construct an element of the new network, starting from the corresponding prospective base node of the new network, then going around the terminal of the non-essential element which would be on the part P not containing the base node, and then returning (back) to the starting prospective base node. (Examples of these constructions, are the elements 1', 2', 3', 4', 5' of the new network, corresponding to the elements 1, 2, 3, 4, 5, of the given network, as shown in Fig. 1, where dashed lines were used for the new elements.)

With this construction we are sure that the new network will also be planar, since it will have no crossings amongst the new elements. Moreover, since exactly one element, and exactly one component, of the new network was constructed for each element, and for each component, of the given network, we are sure that the number of elements in both networks, and the number of components in both networks, will be (respectively) the same. (It shall be found convenient in carrying out the above construction to give corresponding elements, and to give corresponding components, of the two networks, the same numbers.) Furthermore, since each node, except the base node, of a given component, will be enclosed in a single opening (mesh) of the new network, we see that the number of independent meshes of the new network will be equal to the number of independent nodes of the given network (and it shall be convenient to arrange things in such a way as to give a node of one network enclosed by a mesh of the other network the same numbers). [For further reading we recommend: D. Hilbert & Cohn-Vossen, Anschauliche Geometrie, Berlin (1932); of which there is now an english translation of Chelsea Publishing Co., N.Y. (1952).]

With the above we have gone a long way in the construction of the dual of a given network. Next we must consider the important question of orientation of the elements and meshes of the two networks, in order to satisfy the relations between the incidence numbers of the two networks. Let us imagine that corresponding elements of the two networks have the same numbers, and that a node of one of the networks has the same number as that of the corresponding mesh of the other network (as we mentioned above). Then the relations between the ineidence numbers which we seek to fulfill can be expressed thus: $(k,n) = [k,n]^*$ and $[k,m] = (k,m)^*$ for all $k=1,2,\ldots,n_e(=N_e)$, $n=1,2,\ldots,n'_n(=N_m)$, and $m=1,2,\ldots,n_m(=N'_n)$.

An important consequence of the above construction is that elements in series in one of the networks will have corresponding elements of the other network connected between the same pair of nodes, and so they will be connected in parallel in the latter, In order to obtain these relations, it shall be convenient to use a consistent and systematic scheme. This may be done by orienting all the meshes of one of the networks in the counter-clockwise direction (say), and then orienting all the meshes of the other network in the other (clockwise) direction. Then if a given typical (essential) element $\underline{k}$ on the mesh $\underline{m}$ of the first network is oriented in the sense of the mesh $\underline{m}$ (so that $[k,m]=+1$), the corresponding element $\underline{k}$ of the second network must be directed away from the node $\underline{m}$ of this network (so that $(k,m)^*=+1$); otherwise it should be oriented towards the node $\underline{m}$ of this network (so that $[k,m]=(k,m)^*=-1$); thus in each case we will have: $[k,m]=(k,m)^*$ for the mesh $\underline{m}$. With this we can also be sure that the incidence number of the given element $\underline{k}$ with any other mesh $\underline{m}'$ of the first network will always be equal to the incidence number of the element $\underline{k}$ with the corresponding node $\underline{m}'$ of the second network; because both change sign together when the mesh $\underline{m}'$ is adjacent to the mesh $\underline{m}$, and in every other case they vanish together. (See Fig. 2.) Moreover, since we oriented all the meshes of the second network in the clockwise direction, the relative orientations of the elements of the second network with respect to its nodes and meshes will be the same as that of the corresponding elements of the first network relative to its nodes and meshes (as shown in Fig. 3). Thus we will always have: $[k,n]^* = (k,n)$ also. [Concerning a non-essential element of one of the networks, and the corresponding element of the other network (which is closed upon itself), we may mention that the corresponding incidence numbers do not appear in Kirchhoff's Laws, so that they are irrelevant.]

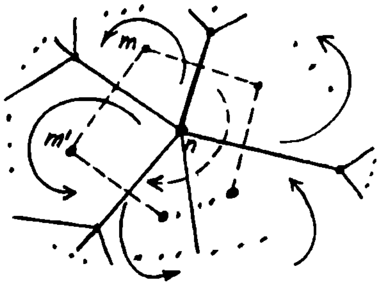


Fig. 2.

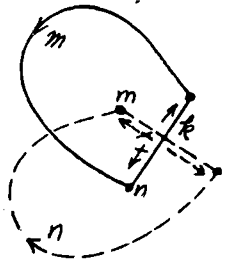


Fig. 3.

As a consequence of the above, we can be sure that Kirchhoff's current laws for one of the networks are interchanged with Kirchhoff's voltage laws for the other network (and vice versa), in a one to one manner, when the currents of one of the networks are interchanged with the voltages of the other.

Finally, in order to determine the dual of the given planar network completely, we must show how the types of elements of the new network can be chosen in such a way that each of its elements will have an equation relating the voltages and currents that will become interchanged with the corresponding equation of the corresponding element of the given network, when all the dual quantities (of the table given above) are interchanged. But this is very easy. If the elements of the given network were chosen to be of the general series type, those of the other network must be of the general parallel (shunt) type, and vice versa; a voltage source passing over into a current source, and vice versa, while a passive element passes over into a passive element of the same type (a resistance passing over into a conductance, an elastance into a capacitance, and an inductance into an invertance (whether self or mutual), and vice versa). Any series branch in one of the networks will then pass over into a parallel (or shunt) branch in the other, with current and voltage sources becoming interchanged, but with passive elements becoming interchanged with passive elements of the same kind.

With this, all the equations of one of the networks pass over, in a one to one manner, into the corresponding equations of the other network, when the dual quantities are interchanged, and so they will be dual networks.

Previously we had already shown that a network with a dual was necessarily planar; and what we have now just shown is that it is sufficient for a network to be planar in order that it have a dual (with respect to the table of dual quantities given in the introduction to this chapter). Actually we have shown more than this, since we have also given explicit instruction to construct the dual of a given planar network.

Example. Consider the network shown in Fig. 4 as a network of elements of the general series type. Then the number of elements is $n_e=7$, the number of nodes is $n_n=4$, the number of components is $n_c=2$, the number of independent nodes is $n_n'=n_n-n_c=2$, and the number of independent meshes is $n_m=n_e-n_n'=5$. Let us number and orient the elements and meshes as shown in the figure (with all the meshes oriented in the counter-clockwise sense). The graph of this network (of elements of the general series type) is shown in solid lines in Fig. 5. And the graph of the dual network (=the dual graph) is shown in the same fig. 5, but in broken (dashed) lines; and the corresponding network of elements of the general parallel (shunt) type is shown in Fig. 6. This network has $N_e=7$

elements of the shunt type, $N_n=7$ nodes, $N_c=2$ components, $N'_n=N_n-N_c=5$ independent nodes, and $N_m=N_e-N'_n=2$ independent meshes. The elements, nodes, and meshes have been numbered and oriented according to the scheme explained above, all the meshes being oriented clockwisely.

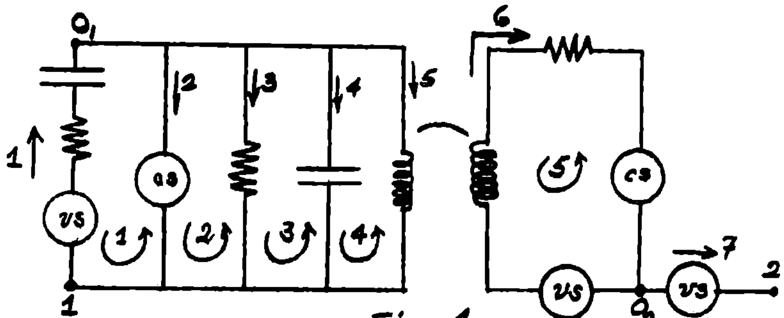


Fig.4.

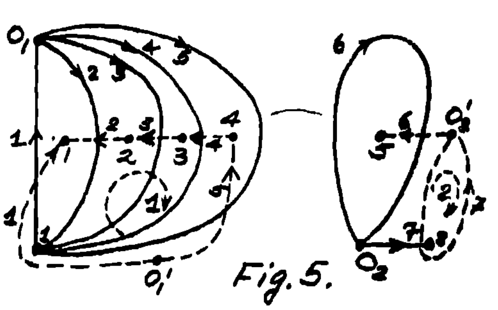
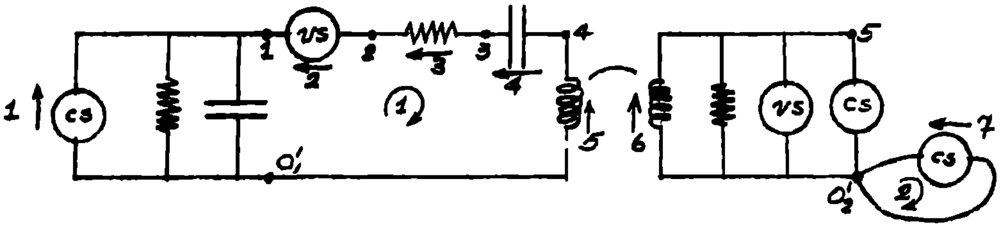


Fig.5.

Fig. 6.



By writing down the complete systems of independent equations for the networks shown in Figs. 4 & 6, it can easily be checked that they become interchanged when all the dual quantities are interchanged.

Problem. Considering the network of Fig. 4 as a network of (nine) elements of the general parallel type, show that its dual, considered as a network of elements of the general series type (of course), is (also) the same network of Fig. 6.

§3. THE SPECIAL DUALITY PRINCIPLE.

In the preceding section we have seen that any given network has a dual with respect to the dual quantities of the table given in the introduction to this chapter if and only if it is planar, and we have given explicit rules for the construction of the dual; and thus the question of duality in this sense is completely settled. The principle embodying these results may be called the (general) duality principle, to distinguish it from another (special) form of duality, which we will consider in this section.

This special form of duality is limited to (planar) networks with no (non-zero) mutual-inductances; and we will call the corresponding principle the special duality principle. With all we established in the preceding section its treatment will be very simple indeed; because most of the contents of §2 is of a combinatorial *nature which does not depend on the types of elements of the networks.*

Thus given any planar network with no (non-zero) mutual-inductances, simply because it is planar we know that there exists a corresponding network such that both networks have the same number of elements (of corresponding kinds), the same number of components, and each has a number of independent nodes equal to the number of independent meshes of the other, which can then be put in a one to one correspondence and numbered and oriented in such a way that corresponding elements have the same numbers, correspondind nodes and meshes have the same numbers, and the incidence numbers of corresponding elements with corresponding nodes and meshes are equal. Thus the equations expressing Kirchhoff's current laws for the independent nodes of one of the networks pass over into the equations expressing Kirchhoff's voltage laws for the independent meshes of the other network (and vice versa) when all the corresponding currents and voltages are interchanged, just because the given network (and its dual) is planar.

Now when the networks have no (non-zero) mutual-inductances, besides, it can easily be recognized that the first of eqs. (1) of §1 (with $L_{kl}=0$ *for* $k \neq l$*) become interchanged with the first of eqs. (3) of §1 (with* $\Gamma_{kl}=0$ *for* $k \neq l$*) when the corresponding quantities (in juxtaposition) in the following table are interchanged:*

series	shunt	series	shunt	series	shunt
V_k	I_k	R_k	G_k	X_L	B_C
I_k	V_k	$S_k=\frac{1}{C_k}$	$\Gamma_k=1/L_k$	X_C	B_L
E_k	$-I_{cs_k}$	L_k	C_k	X_k	B_k
D_k	$-I_{vs_k}$			Z_k	Y_k
J_m	U_m			z_{mn}	y_{mn}

where the last entries in the first column and the whole last column have been added in anticipation of later results. This table differs from that given in the introduction to this chapter in two important aspects. First, the entries $i\omega$ *&* $1/i\omega$ *and* $d()/dt$ *&* $\int()dt$ *of that table are missing in this one; and second, coils and condensers are interchanged according to this table (as is indicated by the interchange of* L_k *with* C_k*, instead of* L_k *with* Γ_k *and* S_k *with* C_k *as in that other table).*

Consequently, the systems of equations of two planar networks without mutual-inductances, one considered as a network of elements of the general series type and the other as a network of elements *of the general parallel (shunt) type, whose elements and independent nodes and meshes can be numbered and oriented and put into a one to one correspondence as mentioned above, will interchange in a one to one manner when corresponding quantities in the above table are interchanged. Such a pair of networks are called (special) dual networks with respect to the special correspondence given in the above table (of this section); and corresponding quantities in this table are called dual quantities in the special sense. We will call the above result the special duality principle.*

The rules to construct the dual of a given planar network without mutual-inductances are the same as in §2, with independent nodes and meshes interchanging, with series branches being interchanged with parallel (shunt) branches, with current and voltage sources interchanged, with resistors passing over into resistors, but except that now coils and condenser are interchanged.

As before, quantities defined by functions of the original dual quantities which are interchanged by the interchange of the original dual quantities are also called dual quantities, but now in the special sense of the correspondence between the original quantities given in the above table. It is easy to show that the corresponding quantities in the last row of the first column, and in the whole last column, of the above table are dual quantities in the special sense; and accordingly they were included in the table in anticipation.

Example. Consider the network without mutual-inductances, shown in Fig. 1, as a network of elements of the general series type. The number of elements is $n_e=7$, the number of nodes is $n_n=4$, the number of components is $n_c=1$ and so the number of independent nodes is $n'_n=n_n-n_c=3$, and the number of independent meshes is $n_m=n_e-n'_n=4$. (In networks without mutual-inductances, there is no point in considering more than one component, since each performs independently of the others and so each may be considered separately.) Let the elements, nodes and meshes, be numbered and oriented as shown in the figure (with all the meshes oriented counter-clockwisely).

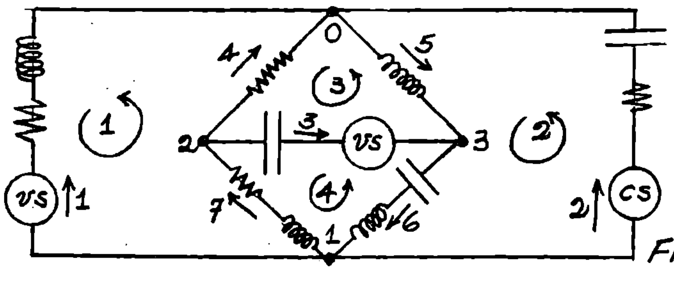


Fig. 1.

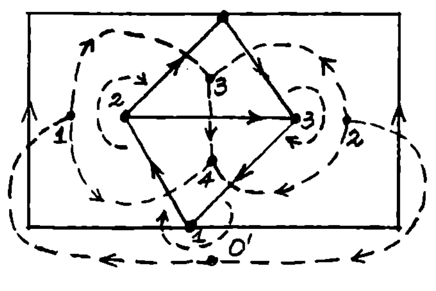


Fig. 2.

The graph of the given network is shown in solid lines, and the graph of its dual is shown in broken lines, in Fig. 2. The dual network (of elements of the general parallel, or shunt, type) is shown in Fig. 3, the elements, nodes, and meshes of which were numbered and oriented as explained in §2, while the elements themselves were constructed according to the table given above in this section, as explained above (the resistor, coil, and vs. in series in the element 1 of the given network, passing over into a resistor, condenser, and cs., respectively, in parallel in the corresponding element 1 of the dual network; the resistor, condenser, and cs. in series in element 2 of the given network, passing over into a resistor, coil, and vs., resp., in parallel in element 2 of the dual, etc.). If the equations of either of these dual networks are established, the equations for the other may be obtained directly by substitution of all the dual quantities in accordance with the table of this section.

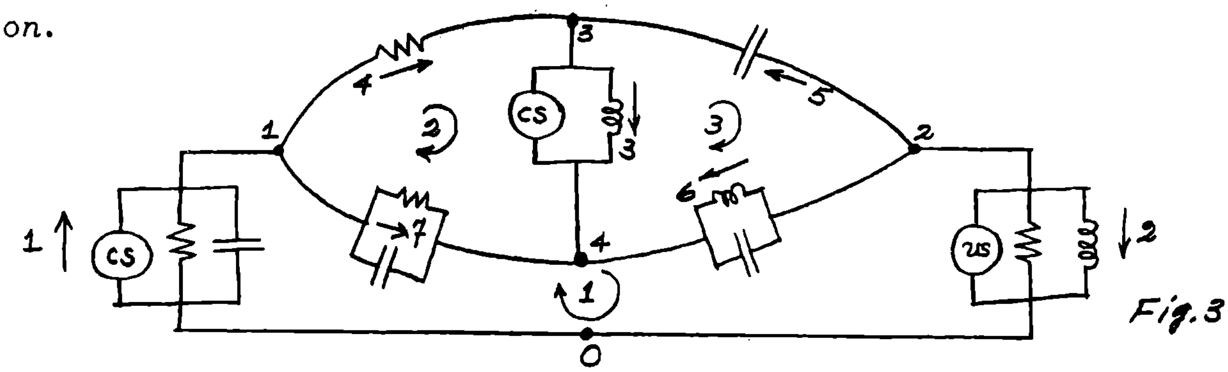


Fig. 3.

Problem 1. Considering the network of Fig. 1 as a network of (fourteen) elements of the general parallel type, show that its dual, considered as a network of elements of the general series type (of course), is still the same network of Fig. 3.

Problem 2. Find the duals of the following networks (Fig. 4), first according to the correspondence given by the table in the introduction to this chapter, then according to the correspondence given by the table of this section, and then compare the results. (For illustrative purposes, consider all the heavy dots shown as the nodes, and assign the numbers and orientations arbitrarily.)

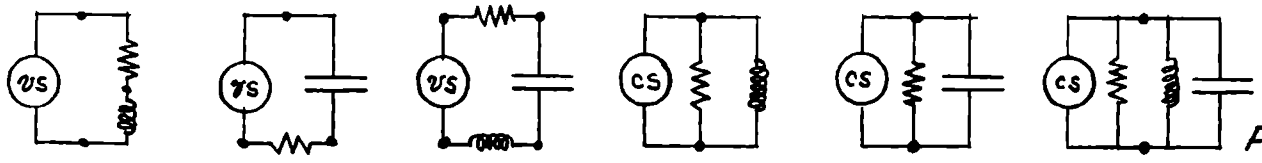


Fig. 4.

Problem 3. Find the duals of the following networks, according to the table of the introduction (since they have none according to the table of this section). (Fig. 5)

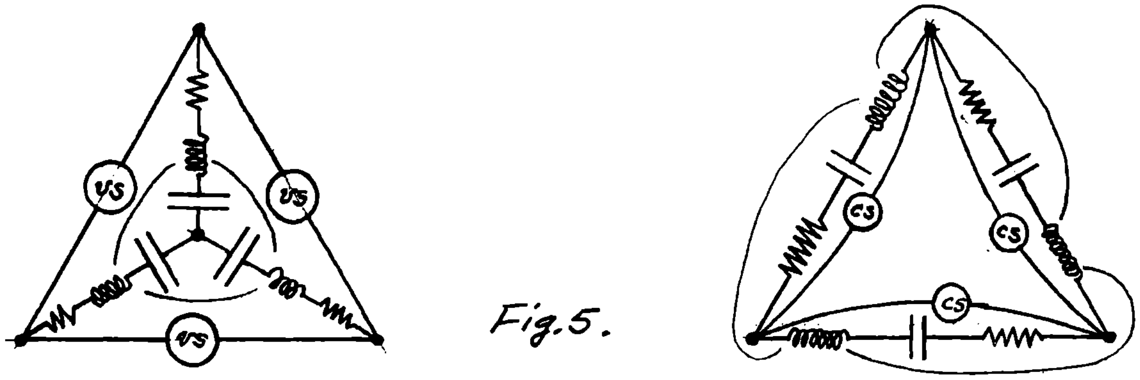


Fig. 5.

Problem 4. Assuming that the mutual-inductances are absent in the networks of Fig. 5, find the duals in the special sense of this section.

Problem 5. Find the dual graphs of the following graphs (of Fig. 6), according to the method and rules given in §2.

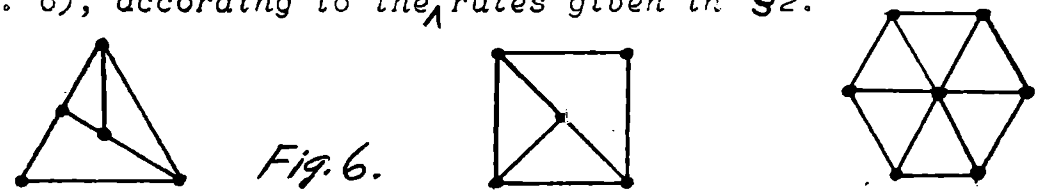


Fig. 6.

§4. INVERSE (OR RECIPROCAL) PASSIVE TWO-TERMINAL STRUCTURES.

By the inverse with respect to a finite resistance $R \neq 0$ (and conductance $G=1/R$) of a passive, two-terminal structure (or box) with mutually accessible terminals, and with an admittance Y (and impedance $Z=1/Y$), we mean a two-terminal box with an impedance Z' (and admittance $Y'=1/Z'$) such that for all frequencies we have:

$$ZZ' = R^2 \quad \text{and} \quad YY' = G^2 \quad \text{and hence:} \quad Z' = R^2 Y \quad \text{and} \quad Y' = G^2 Z. \qquad (1).$$

An important application of the special duality principle is concerned with the construction of such inverses for passive planar two-terminal structures without mutual-inductances. To show how this can be done, consider a given planar two-terminal passive box without mutual-inductances (which may be considered, without loss of generality, to be a single connected part=component, since no magnetic couplings are now allowed), and consider the network formed by connecting a sinusoidal voltage source (of an arbitrary angular frequency ω) across its terminals (as shown in Fig. 1). Now let us construct the dual of this network in the special sense of §3. This dual will clearly ~~consist of~~ be a planar two-terminal passive box, with no mutual-inductances, with a current source connected between its terminals (as shown in Fig. 2 in solid lines, where~~as~~ the original network is repeated in broken lines).

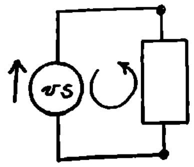


Fig. 1.

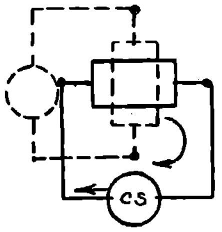


Fig. 2.

Now if the given box is considered as a network of N elements of the general series type with (given) parameters R_k, L_k, S_k ($k = 1,2,\ldots,N$), and its dual is considered as a network of N elements of the general parallel type with parameters G_k, C_k, Γ_k ($k=1,\ldots,N$), and if the latter are chosen such that:

$$G_k=G^2R_k\,,\qquad C_k=G^2L_k\,,\qquad \Gamma_k=G^2S_k\,,\qquad (2)$$

then the equations of the dual network (of Fig. 2), considering the current source to be sinusoidal of the same angular frequency ω, will not only be of the same form as those of the given network (of Fig. 1), with currents and voltages interchanged, but they will also have identical coefficients (as functions of $i\omega$) if the common factors G^2 in all the admittances of the dual network are attached to the voltages. Consequently, upon solving the complex equations of the given network for the ratio of the complex current to the complex voltage drop through the voltage source, and solving the complex equations of the dual network for the ratio of the complex voltage drop (multiplied by G^2) to the complex current through the current source, we obtain exactly the same rational functions of $i\omega$ as the ratios. But the former is precisely the admittance $Y(i\omega)$ of the given box, and the latter is the impedance $Z'(i\omega)$, say, of the dual box (mutiplied by G^2), at the arbitrary angular frequency ω. Thus we will have for all values of the angular frequency ω:

$$Y = G^2\cdot Z' \text{ and hence } Z = R^2\cdot Y',\text{ and so: } Y'=G^2Z \;\&\; Z'=R^2\cdot Y. \qquad (3)$$

When in the above we take $R=1$ & $G=1$, we speak simply of the inverse box of the given box; which will in effect have an impedance equal to the admittance of the given box, and an admittance equal to the impedance of the given box, at all frequencies (numerically, of course).

Problem 1. If A is the inverse box with respect to $R = 1/G$ of a two-terminal planar passive box B without mutual-inductances, show that the structures of Fig. 3 have constant resistance R (and hence constant conductance G) at all frequencies.

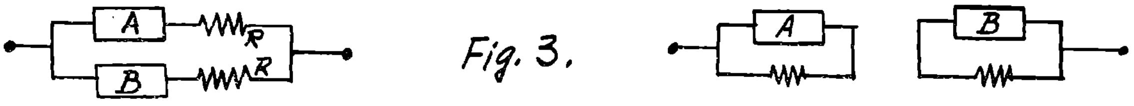


Fig. 3.

§5. GENERALIZATIONS OF THE DUALITY PRINCIPLE.

All the preceding results of this chapter may be immediately generalized to networks of passive box-source elements of the kind considered in Ch. VIII, §3 and §4. This can be recognized by considering the eqs. (1 & 2) of Ch. VIII, §3, of a network of elements of the type shown in Fig. 1 of Ch. VIII, §3, ~~and~~ together with the eqs. (1 & 2) of Ch. VIII, §4, of a network of elements of the type shown in Fig. 1 of Ch. VIII, §4, in place (resp.) of the eqs. (1), and (2), of §1, in the preceding sections. Of course, it will be sufficient to consider that the graphs of the networks of these generalized box-source elements (in which the generalized elements are considered as units replaced by line segments) are planar, and to consider the impedances and admittances of the boxes as indeterminates; and it shall not be necessary that the boxes in themselves be planar structures. And for the corresponding results on duality in the special sense, it will be sufficient (besides) that all the mutual-impedances between the boxes be zero. Naturally, if all the boxes are planar too, each box will have a dual box also, and then all the results can be extended to the interiors of the boxes in all their details; in which case we would obtain the same results which we would obtain in the usual way.

In the same way, all our results on duality may be generalized to the case of networks of active two-terminal structures, by considering the corresponding eqs. (12), and (15), of Ch. VIII, §5.

All the results on duality may also be generalized to networks in general, by considering the original integro-differential equations (3, 4, & 5 of §1, and 4, 5, & 6 of §2) given in Ch. III; for this reason, the derivative- and integral-pairs were included in the table of the introduction to this chapter.

The results are also valid for networks in which all the currents and voltages are exponentially modulated sinusoids. This can be seen by considering the complex (transformed) equations obtained by substituting exponentially modulated sinusoids for all the currents and voltages in the original integro-differential equations and then making use of the isomorphism between exponentially modulated sinusoids and complex numbers given in Ch. IV, §6, in the manner used in Problem 4 of Ch. IV, §6. It may be observed that we need only substitute the general complex frequency $p=\sigma+i\omega$ for $i\omega$ in all the results of the preceding sections in order to obtain the corresponding results for these networks; and for this reason, the pairs: $\underline{p}$ & $\underline{1/p}$, were included in the table of the introduction to this chapter.

CHAPTER XI: NETWORK VECTOR DIAGRAMS.

Essentially a vector diagram of a network in the sinusoidal state (or in which all the currents and voltages are exponentially modulated sinusoids) is a diagram exhibiting all the complex numbers corresponding to the currents and voltages through the elements of the network as directed line segments or vectors. One (rough) way of doing this consists simply in drawing all the complex numbers corresponding to the currents and voltages as vectors on a plane in which a system of rectangular cartesian coordinate axes have arbitrarily been chosen, i.e., in a Gauss-Argand diagram. Usually it is found convenient to draw the currents vectors and the voltage vectors to different scales; and in order to distinguish between these two classes of vectors, two types of arrowheads may be used. It is common practice to use solid (or closed) arrowheads for complex currents and ordinary (or open) arrowheads for complex voltages (as shown in Fig. 1). Another thing that may be done is to split the diagram in two: one for current vectors (=complex currents) and the other for voltage vectors (=complex voltages); but this is seldomly done.

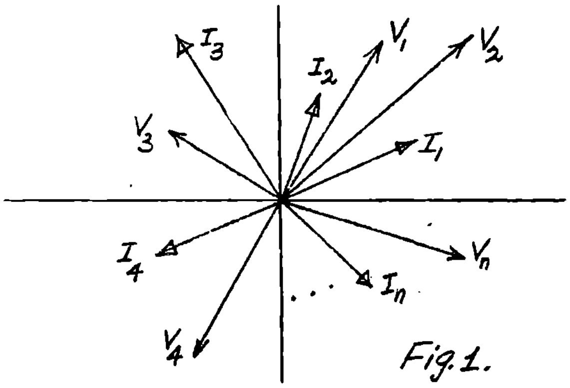


Fig.1.

The angle which a particular vector of the diagram makes with the + side of the ~~x~~ abscissa (or real)-axis is the phase andgle of the corresponding sine function, and since all the vectors of the diagram correspond to sine functions of the same frequency, all their phase angles may be changed (augmented or diminishes) uniformly by a common amount by making a suitable change of time origin. This means that the relative positions of the vectors in the diagram do not depend on the time origin, and that it is only their positions with respect to the coordinate axes that does so; and that the axes may be thrown into any position whatsoever by a suitable choice of the time origin. But in alternating currents (which is essentially a theory of steady states) the origin of the time is irrelevant. Consequently the coordinate axes are usually omitted; it being understood that anyone wishing to be specific can place the system of axes anywhere and be sure that for a suitable time origin they are in the proper place.

§1. VECTOR DIAGRAMS OF KIRCHHOFF'S LAWS.

A better way of giving a vector diagram of a network consists not in drawing all the vectors representing the complex currents and voltages at the origin but, instead, in drawing them in such a way that Kirchhoff's complex current and voltage laws for the nodes and meshes of the network are exhibited explicitly. Since these laws express that certain algebraic sums of complex numbers vanish, they will be exhibited graphically by geometric additions of directed line segments adding up to 0 and thus forming closed polygons.

Thus for a network with a graph as shown in Fig. 1, the vector diagram could be given as shown in Fig. 2, in which Kirchhoff's laws for all the meshes and nodes are clearly exhibited. Thus

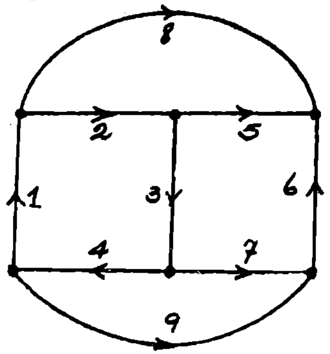


Fig. 1.

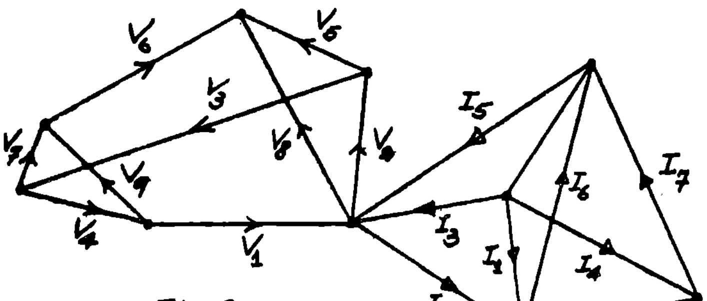


Fig. 2.

Kirchhoff's voltage law for the mesh 125$\overline{67}$4 is clearly exhibited by the closed polygon formed by the vectors V_1, V_2, V_5, $-V_6$, $-V_7$, V_4; and Kirchhoff's current law for the node at which elements 1, 4, & 9 meet, is clearly exhibited by the closed polygon formed by the vectors I_1, I_9, $-I_4$.

In general, the vector diagram of a network with n_m independent meshes and n'_n independent nodes can be obtained by drawing n_m closed polygons with the complex voltage drops (one for each independent mesh, with the complex voltage drops through the elements of the corresponding mesh drawn in the proper order) and n'_n closed polygons with the complex currents (one for each independent node, with the complex currents through the elements meeting at the corresponding node drawn in the proper order). The closed polygon corresponding to Kirchhoff's law for any other mesh or node will then be traceable on this diagram, since it is implied by Kirchhoff's laws for the independent meshes and nodes.

If a certain mesh has only two elements, or if only ~~only~~ two elements meet at a certain node, then the closed polygons of the corresponding Kirchhoff's laws shall be degenerated polygons. In such cases it is convenient to draw only one of the two vectors and to make an allusion to the corresponding Kirchhoff's law by denoting the single vector drawn with the two symbols it represents. Thus for meshes as shown in Fig. 3(a), the corresponding parts of the vector diagram are shown in Fig. 3(b); and for nodes as shown in Fig. 4(a), the corresponding parts of the vector diagram are shown in Fig. 4(b).

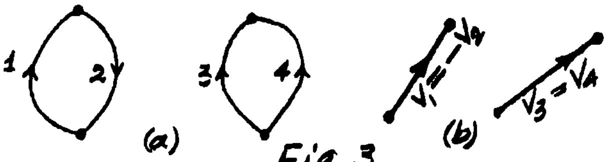


Fig. 3.

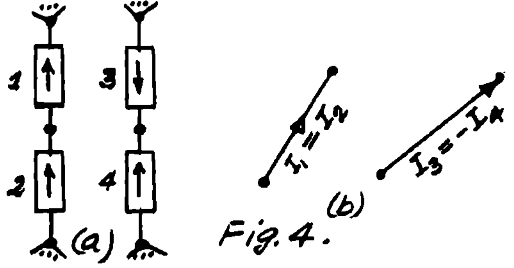


Fig. 4.

§2. VECTOR DIAGRAMS OF THE VOLTAGE EQUATIONS.

For an arbitrary network, besides the equations expressing Kirchhoff's current and voltage laws, we have the equations connecting the voltages with the currents through the elements of the network. For many purposes it will also be convenient to include in the vector diagram of the network all the vectors corresponding to the terms in these equations in such a way that the equations are exhibited graphically by closed polygons. Thus for a network of n_e general series elements in the sinusoidal state, the equations connecting the complex voltages V_k ($k=1,2,\ldots,n_e$) with the complex currents I_k through the elements are the following (cf. eqs. 16 of Ch. V, §1 or eqs. 1, Ch. VIII, §3):

$$V_k + E_k + D_k = \sum_{l=1}^{n_e} Z_{kl} I_l \qquad (k = 1,2,\ldots,n_e). \tag{1}$$

All the vectors corresponding to the terms of these equations may be arranged in closed polygons (one for each equation) resting on the vectors V_k which already appear in the vector diagram in connection with Kirchhoff's voltage laws. To the terminus (or head) of each V_k we may add the corresponding E_k and D_k on one hand; and on the other, we may start from the origin (or tail) of V_k and form the corresponding sum $Z_{k1}I_1 + \ldots + Z_{kn_e}I_{n_e}$ of all the ZI-drops through the typical element $\underline{k}$. This must close the polygon; because according to eqs. (1), these two sums must have the same resultants, as shown in Fig. 1. In this way the details of the voltage drops through the elements of the network are shown graphically, and so are the voltage equations (1). (Of course, when a diagram gets too involved, all this may be done on separate diagrams.)

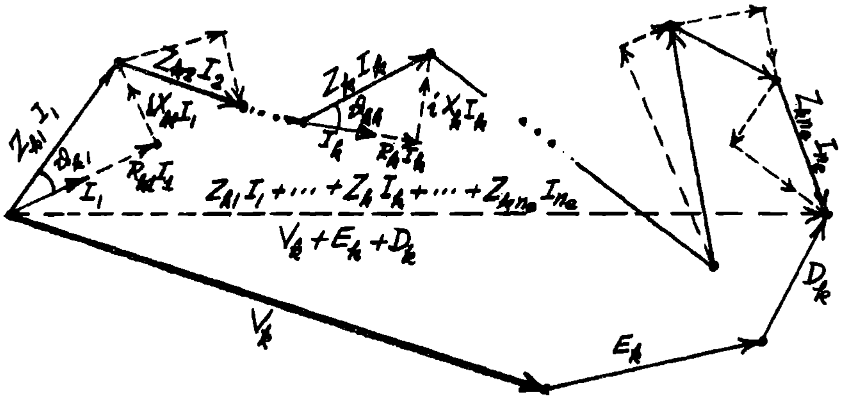


Fig. 1.

The details of each term $Z_{kl}I_l$ may also be exhibited graphically in the diagram. Thus, if $Z_{kl}=R_{kl}+iX_{kl}$, where R_{kl} and X_{kl} are real, we will have: $Z_{kl}I_l = R_{kl}I_l + iX_{kl}I_l$, where $R_{kl}I_l$ is a vector parallel to I_l (or zero if $R_{kl}=0$), and $iX_{kl}I_l$ is a vector at a right angle to (or in quadrature with) the current vector I_l (90° counter-clockwisely ahead of I_l if $X_{kl}>0$, and 90° lagging I_l if $X_{kl}<0$). Consequently, each partial ZI-drop can be expressed as the sum of two vectors, one along the corresponding current vector and the other in quadrature with it, as shown in dashed lines in Fig. 1. Another way of considering this follows by putting $Z_{kl} = |Z_{kl}|\angle\vartheta_{kl}$, where $|Z_{kl}|$ and ϑ_{kl} are real. We will then have: $Z_{kl}I_l = |Z_{kl}|I_l\angle\vartheta_{kl}$, where $|Z_{kl}|I_l$ is a vector along I_l which is $|Z_{kl}|$ times as large, while the unit vector $\angle\vartheta_{kl}$ rotates it by an angle ϑ_{kl} (counter-clockwisely if $\vartheta_{kl}>0$, and clockwisely if $\vartheta_{kl}<0$).

In the preceding paragraph we have allowed each (self- and mutual-)impedance Z_{kl} to be any complex number (with real part not necessarily zero) in order to cover the case of networks with arbitrary boxes. But when the impedances are those of simple <u>RLS-series</u> <u>branches</u> (=resistor, coil, & condenser, in series), all the mutual-impedances Z_{kl} ($k \neq l$) are purely imaginary of the form $i\omega L_{kl}$; and so each mutual-ZI-drop $Z_{kl}I_l = i\omega L_{kl}I_l$ will be a vector in quadrature with the corresponding current vector I_l causing it (90° ahead of I_l if $L_{kl}>0$, and 90° lagging I_l if $L_{kl}<0$, for $\omega>0$). This would leave a single term of the general form in the sum of the ZI-drops for any given <u>k</u>, namely, the self-ZI-drop: $Z_kI_k = R_kI_k + i\omega L_kI_k$.

The above graphical representation of the voltage equations also holds for networks with (all) exponentially modulated sinusoidal currents and voltages, of course; but in this case instead of the pure imaginary $i\omega$ we have the general complex frequency $p = \sigma + i\omega$.

§3. VECTOR DIAGRAMS OF THE CURRENT EQUATIONS.

In the case of a network of n_e general parallel elements, the equations connecting the complex (terminal) currents with the complex voltage drops through the elements of the network are the following (cf. eqs. 6 of Ch. V, §2, for example, or eqs. 1, Ch. VIII, §4):

$$I_k = I_{cs_k} + I_{vs_k} + \sum_{l=1}^{n_e} Y_{kl} V_l \qquad (k = 1, 2, \ldots, n_e). \qquad (1)$$

The detailed structure of each complex (terminal) current (which already appears in the vector diagram of the network in connection with Kirchhoff's current laws for its nodes) can be exhibited graphically on the vector diagram by constructing (in accordance with eqs. 1) the corresponding closed polygon upon each current vector taken as one side, in the manner shown in Fig. 1 for the typical element k.

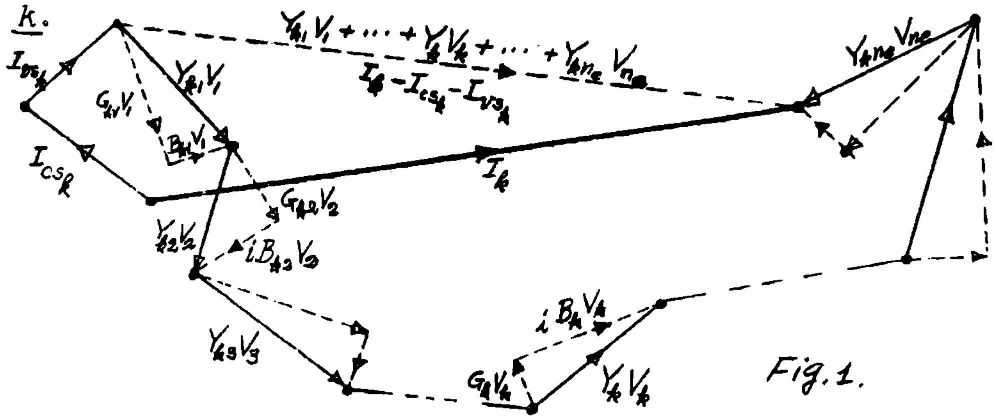


Fig. 1.

In terms of the conductances G_{kl} and the susceptances B_{kl} we have for the admittances $Y_{kl} = G_{kl} + iB_{kl}$; and since the G_{kl} and B_{kl} are real, we see that the contribution of V_l to the (terminal) complex current through the element l, namely, $Y_{kl}V_l = G_{kl}V_l + iB_{kl}V_l$ can be expressed as the sum of two vectors: one along the corresponding voltage drop vector (and G_{kl} times as large), and the other in quadrature with it (and B_{kl} times as large), as shown in dotted lines in Fig. 1. If we put: $Y_{kl} = |Y_{kl}| \angle \vartheta_{kl}$, where $|Y_{kl}|$ and ϑ_{kl} are real, we will also have: $Y_{kl}V_l = |Y_{kl}| V_l \angle \vartheta_{kl}$, which shows that we can also obtain $Y_{kl}V_l$ by first constructing the vector $|Y_{kl}| V_l$ along V_l ($|Y_{kl}|$ times as large as V_l) and then rotating the result by an angle ϑ_{kl} (counter-clockwisely if $\vartheta_{kl} > 0$, and clockwisely if $\vartheta_{kl} < 0$).

In the preceding paragraph we have allowed the admittances to be any complex numbers in order to cover the case of networks with arbitrary boxes. But when the admittances are those of simple GCΓ-parallel branches (=resistor, coil, & condenser, in parallel), all the mutual-admittances Y_{kl} $(k \neq l)$ are purely imaginary of the form $\Gamma_{kl}/i\omega$; and so each mutual-YV-current $Y_{kl}V_l = -i(\Gamma_{kl}/\omega)V_l$ will then be a vector in quadrature with the corresponding voltage drop vector V_l (90° ahead of V_l if $\Gamma_{kl}<0$, and 90° lagging V_l if $\Gamma_{kl}>0$, for $\omega>0$), leaving a single term of the general form in the sum of the YV-currents, for any given $\underline{k}$, namely, the self-YV-current: $Y_kV_k = G_kV_k + iB_kV_k$.

Of course, the above graphical representation of the current equations also holds for networks in which all the currents and voltages are exponentially modulated sinusoids; but in this case we have the general complex frequency $p = \sigma + i\omega$ instead of simply $i\omega$.

§4. THE VECTOR DIAGRAMS OF THE CHARGES AND FLUXES.

Sometimes it is also convenient to include in the vector diagram of a network the complex charges on the condensers, and the complex fluxes through the coils, of the network.

The complex charge Q_k on the condenser (of capacitance C_k and elastance S_k) of the element $\underline{k}$, through which there is a complex current I_{C_k} and a complex voltage drop V_{C_k} (say), is (cf. Ch. V, §1, Prob. 14):

$$Q_k = I_{C_k}/i\omega = C_kV_{C_k} = V_{C_k}/S_k \,. \tag{1}$$

Thus: $|Q_k| = |I_{C_k}|/\omega = C_k|V_{C_k}|$, and: $\operatorname{ang} Q_k = \operatorname{ang} I_{C_k} - 90° = \operatorname{ang} V_{C_k}$.

From this we see that the charge vector (=dircted line segment representing the complex charge) Q_k lags the current vector I_{C_k} by 90° (for $\omega>0$) and it is ω times smaller; and also that Q_k is a vector along the voltage-drop vector V_{C_k}, but C_k times larger (or $S_k = 1/C_k$ times smaller). This is illustrated in Fig. 1.

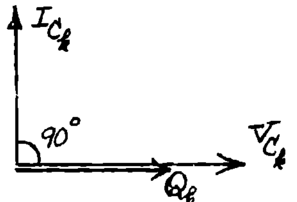


Fig. 1.

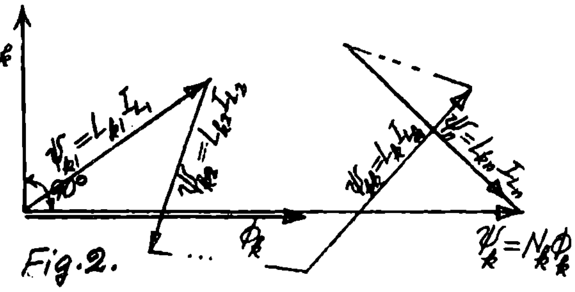


Fig. 2.

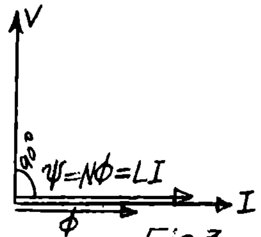


Fig. 3. (idealized)

Now let us consider a (magnetically closed) system of $\underline{n}$ coils, with the self- and mutual-inductances L_{kl}, the reactances $X_{kl} = \omega L_{kl}$, and the turns N_k (for $k \,\&\, l = 1,2,\dots,n$), through which there are complex currents I_{L_k} and complex voltage drops V_{L_k} (all referred to assigned reference directions). The complex flux-linkages Ψ_k, and the complex fluxes Φ_k, through the coils (in the associated reference directions) will be given by (cf. Ch. V, §1, Prob. 15):

$$\Psi_k = N_k\Phi_k = V_{L_k}/i\omega = \sum_{l=1}^{n} L_{kl} I_{L_l} = \sum_{l=1}^{n} X_{kl} I_{L_l} = \sum_{l=1}^{n} \Psi_{kl}, \qquad (2)$$

where the $\Psi_{kl} = L_{kl} I_{L_l}$ is the contribution of the complex current through the coil $\underline{l}$ to the complex flux-linkages Ψ_k associated with the coil $\underline{k}$. From this we see that each flux-linkage vector Ψ_k is along the flux vector Φ_k (but N_k times larger) and lagging the corresponding voltage-drop vector V_{L_k} by 90° (and ω times smaller than it). We also see that each flux-linkage vector Ψ_k is the sum or resultant of all the partial flux-linkage vectors Ψ_{kl} (for $l=1,2,\ldots\ldots,n$), each of which (in turn) is a vector along the corresponding complex current I_{L_l} producing the contribution. This is illustrated in Fig. 2. Of course, when $n=1$ we have the case of a single (magnetically isolated) coil, the corresponding vector diagram of which is shown in Fig. 3.

§5. EXAMPLES.

Example 1. The vector diagrams of the following simple networks are adjoined. (They should now be understood without explanation.)

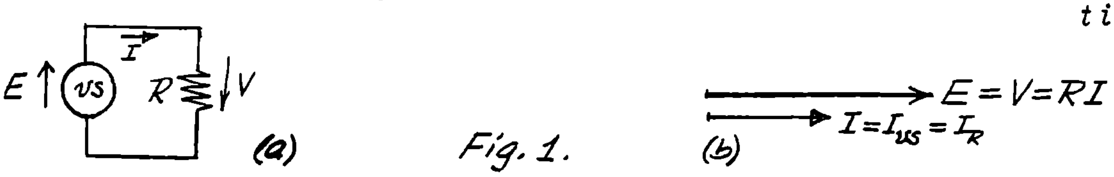


Fig. 1.

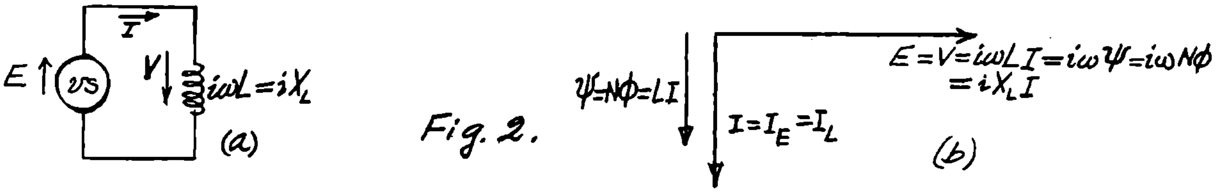


Fig. 2.

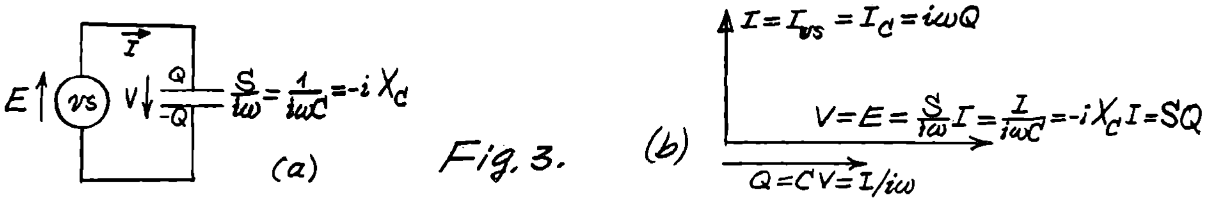


Fig. 3.

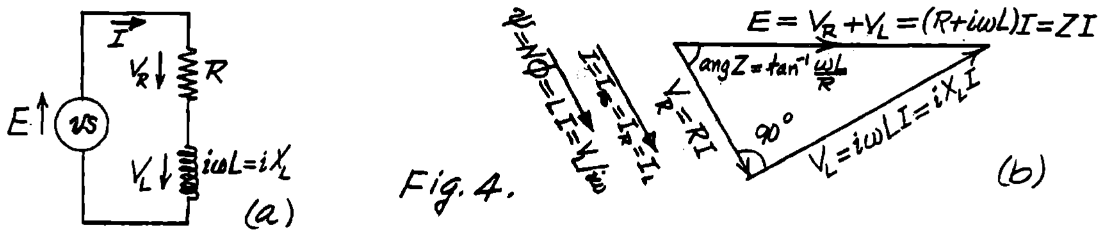


Fig. 4.

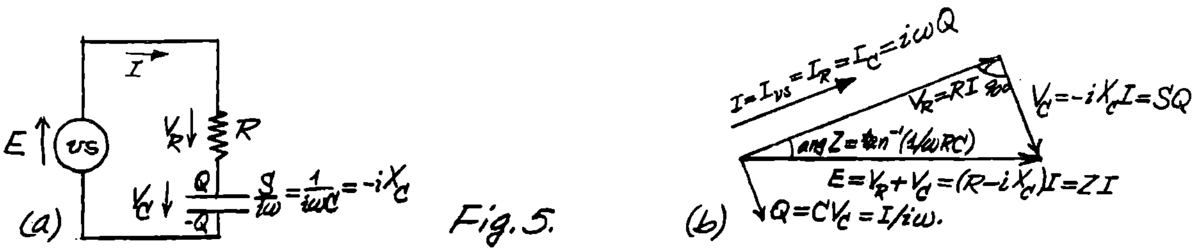


Fig. 5.

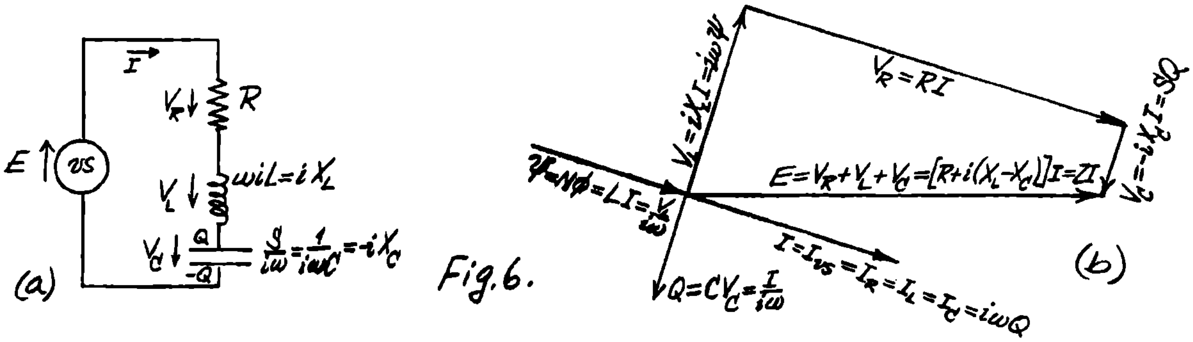


Fig. 6.

Example 2. Consider the network shown in Fig. 7(a), with three elements of the general series type. Kirchhoff's current law for either of the two nodes of the network is exhibited by the closed polygon formed by the three current vectors in (b). The voltage law applied to the meshes (1,2) and (2,3) are indicated in (c) by the equalities $V_1=-V_2$ and $V_2=V_3$, respectively. The voltage equation: $V_1=Z_1I_1-E$, or equivalently: $V_1+E=Z_1I_1$, for the element 1 is exhibited by the closed polygon formed by the vectors: E, V_1, and Z_1I_1 (the details of the latter being exhibited in dotted lines) in (c). The voltage equations: $V_2=R_2I_2+(S_2/i\omega)I_2$ and $V_3=R_3I_3+i\omega L_3I_3$ for the elements 2 and 3, respectively, are exhibited in (c) by the (closed) triangles above and below the vector $V_2=V_3$. The complex charge on the condenser of element 2, and the complex fluxes linking the coils of elements 1 and 3, are shown in (b); The complex charge Q_2 on the condenser is lagging the corresponding complex current I_2 by 90°; but the complex fluxes ϕ_1 and ϕ_3 are in phase with the corresponding complex currents I_1 and I_3, respectively.

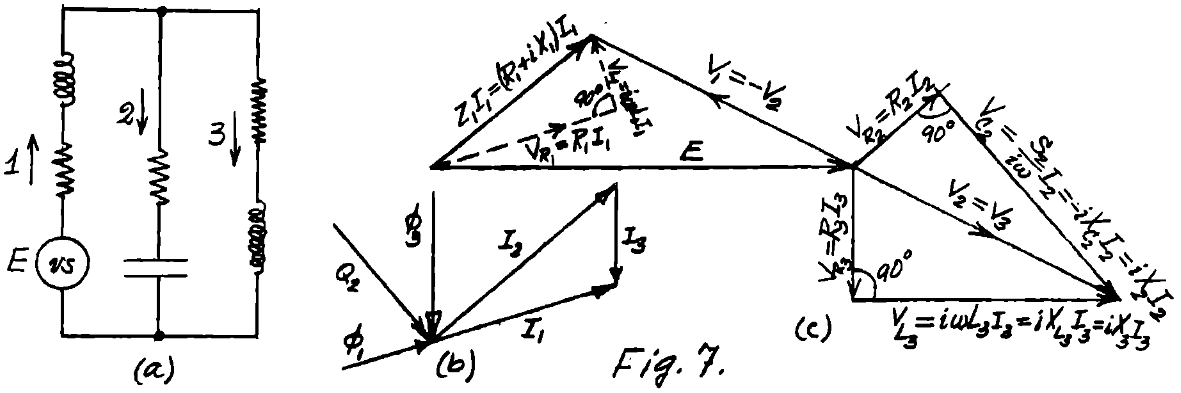


Fig. 7.

Example 3. The vector diagram of the network shown in Fig. 8(a) is given in Fig. 8(b). The diagram was drawn according to the explanations given above.

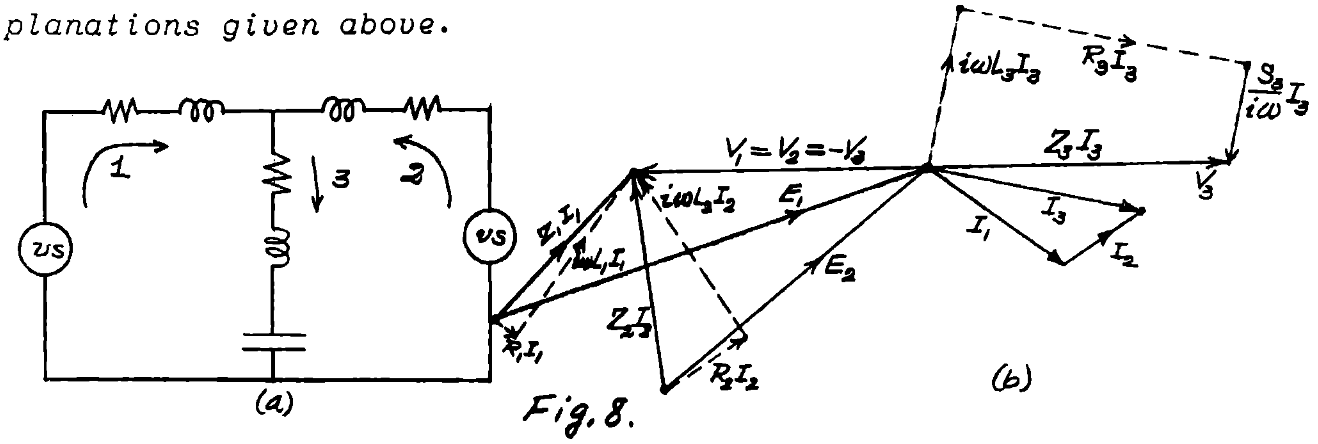


Fig. 8.

Example 4. Consider the network shown in Fig. 9(a). Assume that all the complex currents and voltages through the elements of the network have been found. Assuming that $M>0$, the vectors of the vector diagram (b) of the network were drawn in the following order: E, $I_1=I_3$, $I_2=-I_4$, $Z_1I_1=iX_1I_1+R_1I_1$, $i\omega MI_2$, $V_1=-V_3$, $Z_3I_3=R_3I_3+iX_3I_3$, $Z_2I_2=R_2I_2+iX_2I_2$, $i\omega MI_1$, $Z_4I_4=R_4I_4+iX_4I_4=R_4I_4+i(X_{L_4}-X_{C_4})I_4$, and $V_2=V_4$.

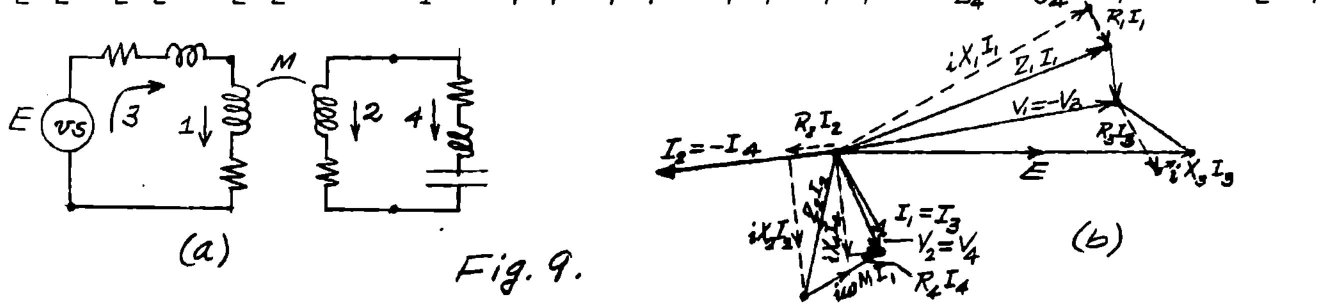


Fig. 9.

Problem 1. Draw the vector diagrams for the networks shown if Fig. 10.

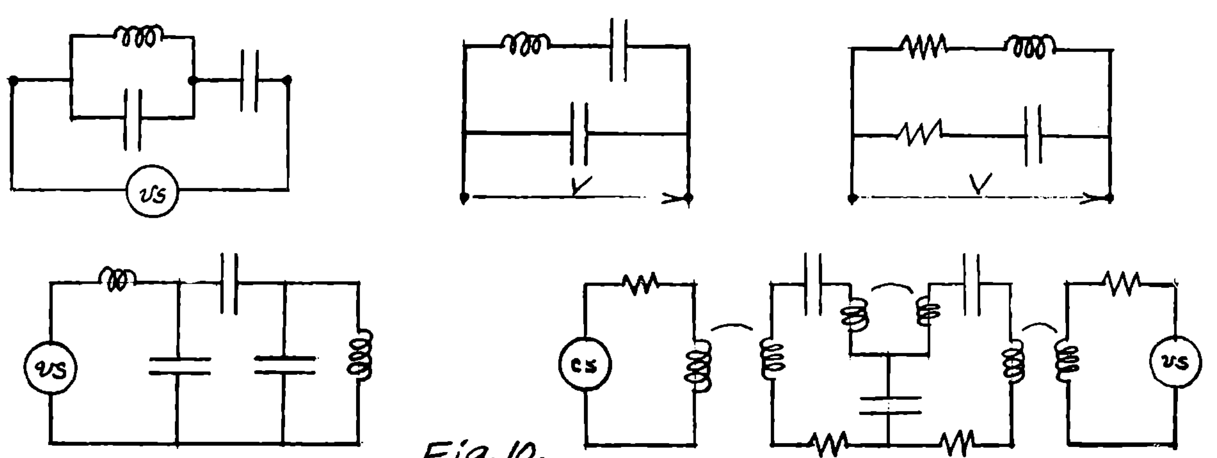


Fig. 10.

Problem 2. Establish the vector diagrams for the networks shown on pp. 381 & 398 of M. I. T.'s Electric Circuits (Wiley, 1943).

Problem 3. Establish the vector diagrams for the networks shown in Fig. 11.

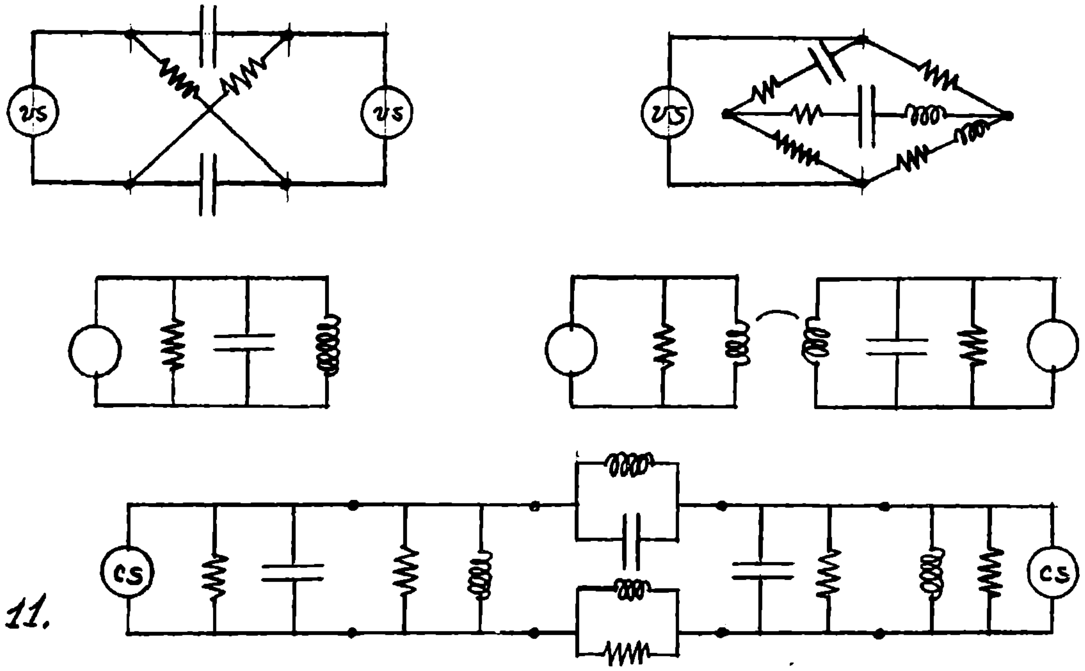


Fig. 11.

Problem 4. Two elements with impedances 10+5i and 5+10i (resp.) are connected in parallel across a common source. If the effective current (=its modulus) through the first of these is 25 amp., what is the effective current through the source. Draw the vector diagram.

Problem 5. Two impedances 10+5i and 5-15i are connected in parallel across a common source through which there is a complex current $60\underline{/0^\circ}$ amp. What are the complex currents through the two impedances. (Hint: Draw the vector diagram to scale.)

Problem 6. A resistance, an inductance, and an elastance are connected in series across a source through which there is a complex current -5+5i and a complex voltage rise (referred to the same reference direction) of 50+100i. If the effective voltage across the condenser is 200 volts, compute the resistance, inductance, and elastance, as well as the complex voltage drops across each of them. Draw the vector diagram to scale.

Problem 7. Two practical coils of impedances 1+2i and 2+i are connected in series across a voltage source of electro-motive force $220 \sin 100\pi t$. What are the sinusoidal voltage drops across each coil and the sinusoidal currents through them. (Hint: First find the complex currents and voltages from the corresponding vector diagram drawn to scale; but in the treatment never mix the time functions with the corresponding complex numbers.)

Problem 8. Consider two sinusoids and the corresponding complex numbers (with $A>0$ & $B>0$):

$$E^* = A\sin(\omega t + a) \longleftrightarrow \underline{E} = \frac{A}{\sqrt{2}}\underline{/a} \qquad I^* = B\sin(\omega t + b) \longleftrightarrow \underline{I} = \frac{B}{\sqrt{2}}\underline{/b}\,.$$

Let $\underline{I}$ be expressed as a sum: $I = I_= + I_\perp$, where $I_=$ is parallel to (or in phase with) $\underline{E}$, and $I_\perp$ is at a right angle (or in quadrature) with $\underline{E}$. By re-transforming this equation back into sinusoids, show that:

$$B\sin(\omega t + b) = B\cos(b-a)\sin(\omega t + a) + B\sin(b-a)\sin(\omega t + a + 90°);$$

which, of course, can easily be obtained (or checked) analytically. The vector $I_=$ is called the in-phase component, and $I_\perp$ is called the quadrature component, of $\underline{I}$ with respect to $\underline{E}$; and the corresponding sine functions $I^*_=$ and $I^*_\perp$ are called the in-phase and quadrature (component) sinusoids of I^* with respect to E^*. Show that: $I_\perp = iI_\parallel \tan(b-a)$.

Problem 9. A voltage source of 220 volts effective value is applied to three impedances in parallel. The first of these takes an effective current of 30 amps. at a lagging angle of 30° with respect to the applied voltage. The second takes 15 amps. at a lagging angle of 45°. The third takes a current in quadrature with the applied voltage, such that the current through the source lags the applied voltage by 5°. Draw a vector diagram to scale of the network and compute the complex currents through the third element & source.

Problem 10. Consider the network shown in Fig. 12, the vector diagram of which is shown in Fig. 13 (in solid lines). Now assume that a condenser is shunted across $\underline{Z}$ as shown in dotted lines. The vector diagram of this new network is shown in dotted lines in Fig. 13. The effect of the condenser is to turn the vector of the current through the voltage source in the counter-clockwise direction. Assuming $\underline{E}$, $r+ix$, and $R+iX$, to be given, by geometrical considerations of the vector diagram, find the capacitance $\underline{C}$ of the condenser so that the current vector $\underline{I'}$ through the voltage source in the new network is in phase with its electromotive force vector $\underline{E}$.

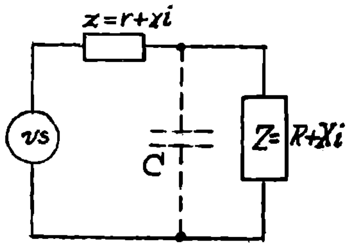


Fig. 12.

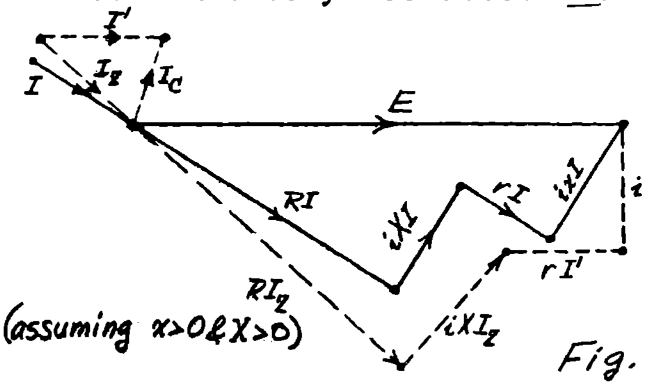


Fig. 13.

Problem 11. Consider a system of *n* practical coils in the sinusoidal state at the angular frequency ω. Let V_k and I_k (for $k=1,2,\ldots,n$) be the complex voltage drop and current through the coil *k*, and let R_k be its resistance, and L_{kl} the mutual-inductance of the coil *k* with the coil *l*, for given reference directions. We will have:

$$V_k = R_k I_k + \sum_{l=1}^{n} i\omega L_{kl} I_l. \tag{1}$$

Show that the corresponding vector diagram for the typical element *k* is of the form shown in Fig. 14, where we have assumed (e.g.): $L_{k1}>0$,

$L_{k2}<0$,

. . . . . .

$L_{kn}>0$,

$\omega>0$.

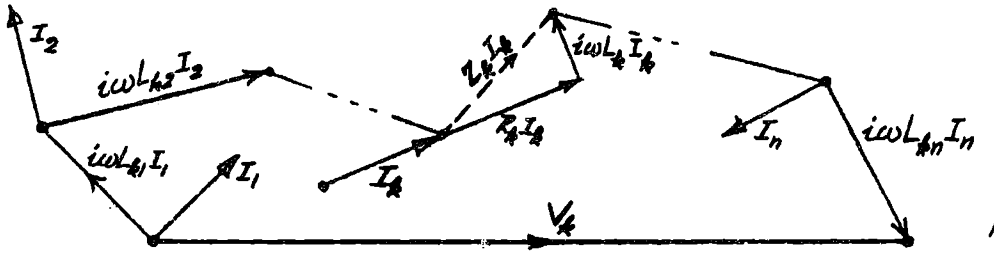


Fig. 14.

§6. VECTOR DIAGRAMS OF THE CANONICAL MESH EQUATIONS.

The general complex canonical mesh equations of an arbitrary network were given in Ch. VIII, §3. Each of these equations can be exhibited graphically on the vector diagram of the network by a closed polygon. Thus the typical equation:

$$I_k = \sum_{m=1}^{n_m} [k,m] J_m \qquad (k = 1, 2, \ldots, n_e), \tag{1}$$

expressing the complex current I_k through the generic element *k* in terms of the complex mesh currents J_m can be represented by constructing on the current vector I_k taken as one side a polygon with the vectors $[k,m]J_m$, as shown in Fig. 1.

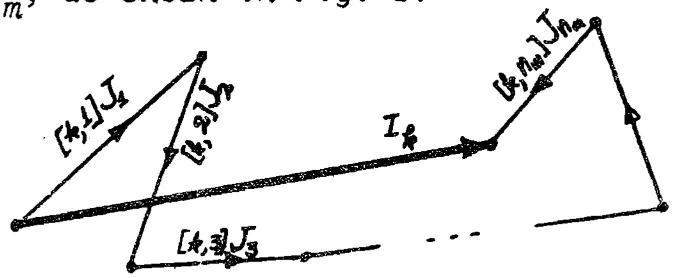


Fig. 1.

Similarly, the typical equation:

$$\sum_{n=1}^{n_m} z_{mn} J_n = \sum_{k=1}^{n_e} [k,m] E_k + \sum_{k=1}^{n_e} [k,m] D_k \qquad (m = 1, 2, \ldots, n_m), \tag{2}$$

expressing that the total complex voltage drop through the passive parts of the generic mesh *m* (due to the complex mesh currents) is equal to the total complex voltage rise through the (current and voltage) sources in this mesh *m*, can be represented by a closed *polygon of the form shown in Fig. 2.*

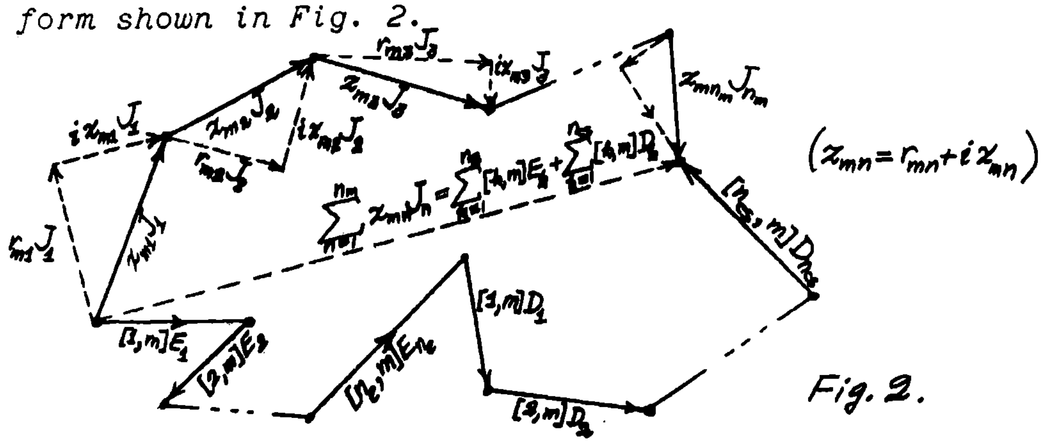


Fig. 2.

§7. VECTOR DIAGRAMS OF THE CANONICAL NODAL EQUATIONS.

The general complex canonical nodal equations of an arbitrary network were given in Ch. VIII, §4. Each of these equations can also be exhibited on the vector diagram of the network by a closed polygon. Thus the typical equation:

$$V_k = \sum_{n=1}^{n_n}(k,n)U_n \qquad (k = 1,2,\dots,n'_n), \tag{1}$$

expressing the complex voltage drop V_k across the generic element <u>k</u> in terms of the complex nodal potential U_n can be represented as a closed polygon constructed on the voltage-drop vector V_k taken as one side with the vectors $(k,n)U_n = \pm U_n$ (or 0), as shown in Fig. 1.

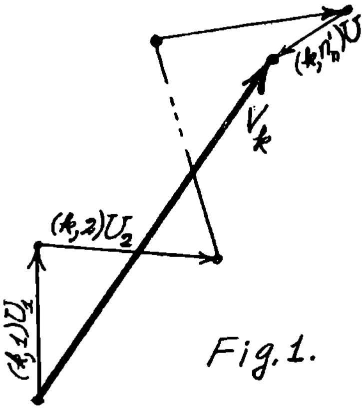


Fig. 1.

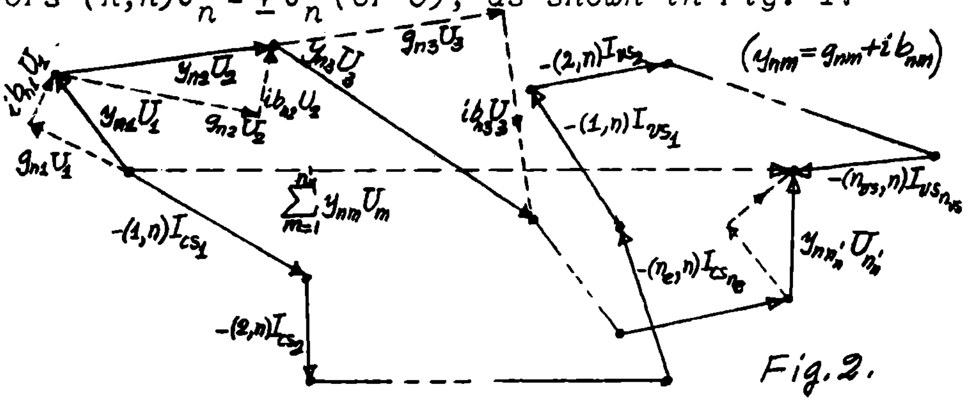


Fig. 2.

In the same way, the typical equation:

$$\sum_{m=1}^{n_n} y_{nm}U_m = -\sum_{k=1}^{n_e}(k,n)I_{cs_k} - \sum_{k=1}^{n_{vs}}(k,n)I_{vs_k} \qquad (n = 1,2,\dots,n'_n), \tag{2}$$

expressing that the total complex current leaving the generic node <u>n</u> through the passive elements (due to the complex nodal potentials) is equal to the total complex current arriving to the node <u>n</u> through the (current and voltage) sources, can be represented by a closed polygon of the form shown in Fig. 2.

<u>Note</u>: When a specific incidence number (k,n)=0, the corresponding null vectors in Figs. 1 & 2 should be considered as absent, of course; and similar remarks hold for §6.

CHAPTER XII: MEAN VALUES AND POWER.

In this chapter we will present the concepts of average or mean values of various orders which are fundamental for the correct understanding and interpretation of measuring instruments, ratings, and specifications, so frequently met with in practice. We will also present the artificially constructed (but very useful) concept of complex (or vector) power and the fundamental theorems on which its use is based.

§1. MEAN VALUES.

The average or (arithmetic) mean value of n numbers $a_1, \dots, a_n$ is defined as a number $\mathfrak{M}(a_1, \dots, a_n)$, frequently denoted simply by $\bar{a}$, such that: $a_1 + \dots + a_n = n \cdot \bar{a}$. If m denotes the smallest of these n numbers, usually denoted by: $\min(a_1, \dots, a_n)$, and M denotes the greatest, usually denoted by: $\max(a_1, \dots, a_n)$, then $n \cdot m \leq a_1 + \dots + a_n \leq n \cdot M$; hence: $m \leq \bar{a} \leq M$, so that $\bar{a}$ is indeed a value in between the smallest and greatest of the given numbers $a_1, a_2, \dots, a_n$ (in case they are real).

Now let us consider an arbitrary real function f of the real variable t, defined in an interval: $a<t<b$. If this interval is divided into n parts of equal sizes: $\Delta t = (b-a)/n$, the mean value of the n values $f(t_k)$ of the function f at the mid-division points $t_k = a + (k-1/2)\Delta t$ $(k=1,2,\dots,n)$ is:

$$\frac{1}{n}\sum_{k=1}^{n} f(t_k) = \frac{1}{n\Delta t}\sum_{k=1}^{n} f(t_k)\cdot\Delta t = \frac{1}{b-a}\sum_{k=1}^{n} f(t_k)\cdot\Delta t,$$

which converges to: $\frac{1}{b-a}\int_a^b f(t)dt$ (as $n \to \infty$) if $f(t)$ has a (proper) integral in the interval (a,b). We will call this limit, usually denoted by $\overline{f(t)}$ or $\mathfrak{M}f(t)$, the mean value of $f(t)$ in the given interval (a,b); and we will define, in general:

$$\mathfrak{M}f(t) = \frac{1}{b-a}\int_a^b f(t)\,dt \qquad (1)$$

to be the (first order) mean value of $f(t)$ in the given interval (a,b) when the integral exists in any given sense. And to complete the definition, when $a=b$ we take the mean value to be 0, and when $a>b$ we take the mean value to be the same as that in (b,a).

Geometrically, the mean value of $f(t)$ in the interval (a,b) can be visualized as the height of a rectangle of base $(b-a)$ with an area equal to the "net area below the curve $y=f(t)$", the area below the t-axis being counted as negative, of course.

The mean value of a given function may depend on the interval in which it is taken, so that the mean value in the interval (a,b) could more properly be denoted with $\mathfrak{M}_a^b$; *but when the interval is clear from the context we will take the liberty of simply using* $\mathfrak{M}$.

Given $f(t)$ *in* (a,b), *and* $t=g(u)$ *in* (a',b'), *such that:* $a=g(a')$ *and* $b=g(b')$, *we will have:* $f(t) = f[g(u)] = F(u)$, *say, for all* $t=g(u)$ *in* (a,b). *However, since a~~n~~ uniform sub-division of the interval* (a,b) *in equal parts shall not necessarily correspond to a~~n~~ uniform sub-division of the interval* (a',b'), *the mean value of* $f(t)$ *in* (a,b) *shall not necessarily be the same as the mean value of* $F(u)$ *in* (a',b'). *The latter of these may be called the mean value of* $f(t)$ *with respect to* $\underline{u}$; *and on account of the above, one should distinguish (with a suitable notation) between mean values of equal functions taken with respect to distinct variables. Fortunately, here we will be concerned only with mean values taken with respect to a given argument (time), so that the distinction need not be stressed.*

Problem 1. *If* $f(t)$ *is a constant* $\underline{A}$ *(say) throughout the interval* (a,b), *show that:* $\mathfrak{M}f(t) = A$. *In particular:* $\mathfrak{M}0 = 0$.

Problem 2. *If* $f(t)$ *is a constant* $\underline{A}$ *in the left (or right) half of the interval* (a,b) *and zero in the other half, show that:* $\mathfrak{M}f(t) = \frac{A}{2}$.

Problem 3. *If* $m \leq f(t) \leq M$ *in* (a,b), *show that:* $m \leq \mathfrak{M}f(t) \leq M$.

Problem 4. *If* $f(t)$ *and* $g(t)$ *have mean values in* (a,b) *and* $\underline{A}$ & $\underline{B}$ *are constants, show that:* $\mathfrak{M}[A\cdot f(t) + B\cdot g(t)] = A\cdot\mathfrak{M}f(t) + B\cdot\mathfrak{M}g(t)$. *Then, by complete induction on* $\underline{n}$, *show that if* $f_k(t)$ $(k=1,2,\ldots,n)$ *have mean values in the same interval* (a,b), *and the* A_k *are constant:*

$$\mathfrak{M}\sum_{k=1}^{n} A_k f_k(t) = \sum_{k=1}^{n} A_k\cdot\mathfrak{M}f_k(t). \tag{2}$$

Problem 5. *If* $f(t)$ *has a mean value in the interval* (a,b) *and* $a<c<b$, *show that:*

$$\mathfrak{M}_a^b f(t) = \frac{c-a}{b-a}\,\mathfrak{M}_a^c f(t) + \frac{b-c}{b-a}\,\mathfrak{M}_c^b f(t). \tag{3}$$

Then, by complete induction on $\underline{n}$, *show that if* $a=a_0<a_1<a_2<\ldots<a_n=b$, *we have:*

$$\mathfrak{M}_a^b f(t) = \sum_{m=1}^{n} \frac{a_m - a_{m-1}}{b-a}\,\mathfrak{M}_{a_{m-1}}^{a_m} f(t). \tag{3}$$

Problem 6. *Show that if the interval* (a,b) *is divided into* $\underline{n}$ *sub-intervals of sizes* Δt_k *(not necessarily equal), and a value* t_k *is arbitrarily chosen in the corresponding sub-interval, and if each value* $f(t_k)$ *is taken a number of times proportional to the size* Δt_k *of the corresponding interval, their mean value shall also converge to* $\int_a^b f(t)\,dt/(b-a)$, *as the greatest* $|\Delta t_k| \to 0$. *(See E. B. Wilson, Advanced Calculus, p. 333.)*

The second order mean value, or root-mean-square (abbreviated rms) value, of n real numbers: $a_1, a_2, \dots, a_n$, is defined as a number $\mathcal{M}_2(a_1,\dots,a_n)$, frequently denoted simply by $\overline{\overline{a}}$, or $\mathcal{N}(a_1,\dots,a_n)$, such that: $n\cdot\overline{\overline{a}}^2 = a_1^2 + \dots + a_n^2$. This second order mean value is also called (by electrical engineers) the effective value of the given n numbers. In the case of a real function $f(t)$ defined in a given interval $a<t<b$, with a square integrable in this interval, the second order mean value, root-mean square (rms-) value, or effective value, of $f(t)$ in (a,b) will be defined to be the quantity:

$$\mathcal{M}_2 f(t) = \mathcal{N} f(t) = \sqrt{\frac{1}{b-a}\int_a^b [f(t)]^2 dt} = \sqrt{\mathcal{M}[f(t)]^2}, \qquad (4)$$

the positive square root being taken. This quantity may be considered as the limit of the second order mean, rms-, or effective, value of n values of the function f at the mid-division points of a homogeneous sub-division of the given interval into n parts of equal sizes, as $n \to \infty$. This quantity is also denoted $\overline{\overline{f(t)}}$ sometimes. Of course, it depends (in general) on the interval (a,b) in which it is taken; so that it could be denoted more properly by using $\mathcal{N}_a^b$ in case it is relevant to mention the interval explicitly.

The r^{th} order mean value of $f(t)$ in $a<t<b$ is defined (when $r \neq 0$) to be the quantity:

$$\mathcal{M}_r f(t) = \sqrt[r]{\mathcal{M}[f(t)]^r} = \sqrt[r]{\frac{1}{b-a}\int_a^b [f(t)]^r dt}, \qquad (5)$$

when these expressions make sense; but we shall not have the opportunity of using other than the first and second order means, here.

Problem 7. Show that the rms-value of a constant A is the absolute value of this constant: $\mathcal{N} A = |A|$. In particular: $\mathcal{N} 0 = 0$.

Problem 8. Show that: $\mathcal{N}[A\cdot f(t)] = |A|\mathcal{N} f(t)$, if A is a constant.

Problem 9. If $|f(t)| \leq M$ (a fixed number), show that: $0 \leq \overline{\overline{f(t)}} \leq M$.

Problem 10. Show that: $[\mathcal{N}(f+g)]^2 = (\mathcal{N}f)^2 + (\mathcal{N}g)^2 \pm 2\mathcal{M}(fg) \geq 0$, and so: $\mathcal{M}(fg) \leq \frac{1}{2}(\mathcal{N}f)^2 + \frac{1}{2}(\mathcal{N}g)^2 = \frac{1}{2}\mathcal{M}f^2 + \frac{1}{2}\mathcal{M}g^2$.

§2. THE CASE OF SINE FUNCTIONS OF THE SAME FREQUENCY.

Consider a sine function $A\sin(\omega t+\alpha)$ of angular frequency $\omega>0$ and period $T = 2\pi/\omega$. Its mean value in the interval $a<t<b$ is:

$$\mathcal{M}[A\sin(\omega t+\alpha)] = \frac{1}{b-a}\int_a^b A\sin(\omega t+\alpha)\,dt = \frac{-A}{\omega(b-a)}\cos(\omega t+\alpha)\Big|_a^b \qquad (1).$$

$$= \frac{A}{\omega(b-a)}\left[\cos(a\omega+\alpha) - \cos(b\omega+\alpha)\right] = \frac{2A}{\omega(b-a)}\sin\frac{(b-a)\omega}{2}\cdot\sin\frac{(a+b)\omega+2\alpha}{2}.$$

From this we see that:

$$\left|\mathfrak{M}[A\cdot\sin(\omega t+\alpha)]\right| \leqq \frac{2|A|}{(b-a)} \rightarrow 0 \quad \text{as} \quad (b-a)\longrightarrow\infty . \qquad (2)$$

Hence, as the size (b-a) of the interval (a,b) increases indefinitely, the mean value of an arbitrary sine function of angular frequency $\omega > 0$ tends to the limit 0; which, incidentally, is also the mean value of the sine function for any interval with a size equal to an integral multiple of periods: $b-a = nT = 2\pi n/\omega$, since then: $\sin\frac{(b-a)\omega}{2} = 0$.

Problem 1. Using: $A\cos(\omega t+\alpha) = A\sin(\omega t+\alpha+90°)$, show that:

$$\mathfrak{M}[A\cos(\omega t+\alpha)] = \frac{2A}{\omega(b-a)}\sin\frac{(b-a)\omega}{2}\cos\frac{(a+b)\omega+2\alpha}{2}. \qquad (3)$$

Now let us consider the mean value of the product of two sine functions of the same angular frequency $\omega>0$, for the interval $a<t<b$. We have:

$$A\sin(\omega t+\alpha).\,B\sin(\omega t+\beta) = \frac{AB}{2}\left[\cos(\alpha-\beta) - \cos(2\omega t+\alpha+\beta)\right]. \qquad (4^{o})$$

Hence (by eq. 2 of §1):

$$\mathfrak{M}[A\sin(\omega t+\alpha).\,B\sin(\omega t+\beta)] = \frac{AB}{2}\left[\mathfrak{M}\cos(\alpha-\beta) - \mathfrak{M}\cos(2\omega t+\alpha+\beta)\right]$$
$$= \frac{AB}{2}\cos(\alpha-\beta) - \frac{AB/\omega}{2(b-a)}\sin(b-a)\omega\cdot\cos[(a+b)\omega+\alpha+\beta] \longrightarrow \frac{AB}{2}\cos(\alpha-\beta) \qquad (4)$$

as $(b-a)\longrightarrow\infty$ (by the results of Prob. 1, §1, and Prob. 1 above).

From this we see that as the size (b-a) of the interval (a,b) increases indefinitely, the mean value of the product of two sine functions of the same angular frequency $\omega>0$ tends to the half of the product of their amplitudes multiplied by the cosine of the angle between them, as limit; which is also the mean value of the product taken in any interval with a size of any integral multiple of half-periods: $b-a = nT/2 = n\pi/\omega$; since then $\sin(b-a)\omega = \sin n\pi = 0$, and so the second term in (4) is 0, which leaves only the first term, namely: $(AB/2)\cos(\alpha-\beta)$, for the mean value. Notice that this limit is independent of ω.

In particular, when the two sine functions are the same, we have for the mean value of their square in the interval $a<t<b$:

$$\mathfrak{M}[A^2\sin^2(\omega t+\alpha)] = \frac{A^2}{2} - \frac{A^2/\omega}{2(b-a)}\sin(b-a)\omega\cdot\cos[(a+b)\omega+2\alpha],$$

which tends to $A^2/2$, as limit, as $(b-a)\longrightarrow\infty$; this is also the mean value taken for any integral multiple of half-periods. The positive square roots of these quantities are the corresponding second order mean, rms-, or effective, values of the sine function; hence:

$$\mathfrak{N}[A\sin(\omega t+\alpha)] = (|A|/\sqrt{2})\cdot\left\{1 - \sin(b-a)\omega\cdot\cos[(a+b)\omega+2\alpha]/\omega(b-a)\right\}^{1/2} \longrightarrow |A|/\sqrt{2}. \qquad (5)$$

Thus as the size (b-a) of the interval (a,b) increases indefinitely, the rms-value, or effective value, of a sine function tends to its amplitude divided by $\sqrt{2}$ as limit, independent of its phase angle and frequency; and this quantity is also the rms- (or effective) value of the sine function taken for any integral multiple of half-periods (as can be seen by eq. 5). It is recalled that this was the reason why we took the isomorphism between sine functions of a given frequency and complex numbers in the form:

$$A\sin(\omega t+\alpha) \longleftrightarrow \frac{A}{\sqrt{2}}\underline{/\alpha}\ , \qquad (6)$$

in order that the modulus of the complex number corresponding to a sine function would be equal to this rms- (or effective) value. (See the note in Ch. IV, §5, p. 104.)

Now when the moving part of the indicating mechanism of an a-c instrument has any substantial inertia (and a natural frequency substantially smaller than that of the sinusoidal currents and voltages operating it) the moving part of the instrument shall not be able to follow the fluctuating torque acting on it, and it shall then settle down in a fixed position corresponding to the mean value of the torque taken in a large interval of time (meaning a few seconds in practice, for the usual frequencies, which is just about the time it takes to look at the instrument after connecting it). As a consequence, if an instrument of this kind were made to have a driving torque on its moving part proportional to the current or voltage acting on it (as in a d-c ammeter or voltmeter), it would always settle at a zero deflection for sinusoidal currents and voltages; but if the instrument is made to have a driving torque on its moving part proportional to the square of the applied current or voltage, it shall settle at a fixed deflection proportional to the limit mean value of the square (= the square of the effective value) of the sinusoidal current or voltage; however, the scales on these instruments are usually marked off proportionally to the square roots of the deflections, so that the effective (or rms-) values may be read off directly. Likewise, an instrument made to have a driving torque on its moving part proportional to the product of a current and a voltage acting on it (which then requires two sets of terminals of course, as in a wattmeter) shall settle down at a fixed deflection proportional to the limiting mean value (4) of their product, in the case of sinusoidal currents and voltages; and the scales on these instruments are usually marked so as to give this reading directly.

For further reading on the subject of electrical measuring instruments, we recommend the following books: Erickson & Bryant, Electrical Engineering, (Wiley, 1952), Ch. 10; Page & Adams, Principles of Electricity (van Nostrand, 1931), 115 (& 64).

Problem 2. Show that the error in taking the limit $\frac{AB}{2}\cos(\alpha-\beta)$ for the mean value $\mathfrak{M}_a^b[A\sin(\omega t+\alpha)\,.\,B\sin(\omega t+\beta)]$ in the interval (a,b) is less than $|AB|/4\pi n<|AB|/12n$ (in absolute value), where n is the (largest) number of periods $T=2\pi/\omega$ in the interval (a,b). From this show that the percentage error $\leqq 1/2\pi n\cos(\alpha-\beta)$, relative to the limit mean value. Thus for a $\cos(\alpha-\beta)=0.8$ (say), this shall be less than 1% after 20 periods, which is just a third of a second for 60 cycles/sec.

Problem 3. Show that the error in taking $|A|/\sqrt{2}$ for the effective value of $A\sin(\omega t+\alpha)$ in the interval (a,b) is less than:

$$\frac{|A|}{\sqrt{2}}\left[\sqrt{1+\frac{1}{\omega(b-a)}}-1\right]\leqq\frac{|A|}{\sqrt{2}}\left[\sqrt{1+1/2\pi n}-1\right]\approx\frac{|A|}{4\sqrt{2}\pi n}\leqq\frac{|A|}{17n},$$

in absolute value, where n has the same meaning as in Prob. 2; thus (relative to $|A|/\sqrt{2}$) we have approximately: $|\%\text{ error}|\leqq 1/4\pi n<1/12n$.

§3. MEAN POWER.

We have seen in Ch. I, §7, that the power (=rate at which energy is) absorbed by (or delivered to) a two-terminal element with a voltage drop $V^*(t)$ in the reference direction for the current $I^*(t)$ through it is:

$$W(t) = V^*(t)\cdot I^*(t). \tag{1}$$

If instead of the voltage drop, the voltage rise $E^*(t)$ in the reference direction for the current is used, then the result: $E^*(t)\cdot I^*(t)$ will be the power (=rate at which energy is) supplied (or delivered) by the element.

Now in a network in the sinusoidal state, all the voltages and currents will be sine functions of the same frequency. Consider a two-terminal element of such a network with a voltage drop $V^*(t) = A\sin(\omega t+\alpha)$ and a current $I^*(t) = B\sin(\omega t+\beta)$ through it. Let $V = (A/\sqrt{2})\underline{/\alpha}$ and $I = (B/\sqrt{2})\underline{/\beta}$ be the corresponding complex (voltage and current) numbers. An ideal voltmeter (causing no disturbance in the behavior of the network when) connected across the element would measure $|A|/\sqrt{2} = |V|$ which is the modulus (or absolute value) of the complex voltage, and also the rms-, or effective, value of the sinusoidal voltage taken for an infinite interval or for any integral number of periods. In the future we will refer to this simply as the effective voltage, without any explicit interval being mentioned. Likewise, an ideal ammeter (causing no disturbance when) inserted in (series with) the element would measure $|B|/\sqrt{2} = |I|$ which is the modulus (or absolute value) of the complex current, and also the rms-, or effective, value of the sinusoidal current taken for an infinite interval (or for any integral number of periods). We will refer to this simply as the effective current, without any interval being explicitly mentioned, in the future.

The power absorbed by the element will be given (at each instant) by eq. (4º) of §2. This instantaneous power is no longer a sine function, unless the constant term in the second member is zero; and even then it would not be a sine function of the same frequency as that of the voltage and current (but of double their frequency). Hence the fundamental isomorphism between sine functions of a given frequency and complex numbers cannot be used to obtain the instantaneous power, since the latter is not even a member of the class of sine functions of the given frequency. However, the instantaneous power is not the concept of most interest (in practice), but its mean value taken in a large (strictly infinite) interval (which is the same as that taken for any integral number of periods). This quantity, given by eq. (4) of §2, namely: $\overline{W(t)}=(AB/2)\cos(\alpha-\beta)$, would be that measured by an ideal wattmeter properly connected to the element (whose insertion is assumed to have no effects on the currents and voltages of the network). In the future we will refer to this quantity simply as the mean power (and in practice, simply as the power) absorbed by the element, without any explicit mention of an interval. Denoting it simply by W, we will then have:

$$W=\frac{AB}{2}\cos(\alpha-\beta)=\frac{A}{\sqrt{2}}\frac{B}{\sqrt{2}}\cos(\alpha-\beta)=|V|\cdot|I|\cos(\alpha-\beta)=|V|\cdot|I|\cos\varphi, \qquad (2)$$

where we have put: $\varphi=\alpha-\beta=\text{ang}\,V-\text{ang}\,I$ for the angle between the current and the voltage, assuming that $A\geqq 0$ & $B\geqq 0$ (or $AB\geqq 0$), which can always be done.

In a d-c network, in which all the currents and voltages are (considered) constant, the power taken by an element is equal to the product of the current and voltage drop through it; thus the reading of a wattmeter would be equal to the product of the readings of an ammeter and a voltmeter properly connected to the element. In the light of this, a long time ago (but not now), it was considered rather incomprehensive that in a network in the sinusoidal state, the reading of a wattmeter (=mean power) was not necessarily equal to the product of the reading of an ammeter (=effective current) by the reading of a voltmeter (=effective voltage) when these instruments were properly connected to an element of the network (as a fact of experience). The useful factor by which the product of the (effective) current and voltage had to be multiplied to give the (mean) power was then introduced and called the power factor of the element (and it was a fact of experience that it was never greater than 1, in absolute value). Of course, all this is now explained by eq. (2), according to which the power factor of an element is the cosine of the angle φ between the current and voltage drop through it, which is never greater than 1 (in absolute value) since the angle φ is always real.

The power factor of an element is frequently denoted by pf., or PF., so that eq. (2) for the mean power W taken by the element may be written in the following form:

$$W = |V|\cdot|I|\cdot pf., \qquad \text{or} \qquad pf. = W/|V|\,|I|. \qquad (3)$$

When $pf.>0$, the element consumes energy on the average, since the mean power absorbed by the element is positive; and when $pf.<0$, it supplies energy on the average, since then the mean power absorbed by the element is negative, which means that the mean power supplied by the element is positive; assuming that $W \neq 0$, of course.

Problem 1. If $V^* = A\sin(\omega t+\alpha)$ and $I^* = B\sin(\omega t+\beta)$, and $AB<0$, show that $\varphi=\alpha-\beta\pm 180°$, and: $pf. = \cos\varphi = \cos(\alpha-\beta\pm 180°) = -\cos(\alpha-\beta)$. [Hint: If $A<0$, then $V^* = A\sin(\omega t+\alpha) = |A|\sin(\omega t+\alpha+180°)$, etc.]

Problem 2. If $V=(A/\sqrt{2})\angle\alpha = a + a'i$ and $I=(B/\sqrt{2})\angle\beta = b + b'i$, show that: $W=(AB/2)\cos(\alpha-\beta) = ab + a'b'$. This is the ordinary scalar product $V \circ I$ (say) of the vectors V and I in the complex plane, and so the mean power absorbed by an element is given by the scalar product of the complex voltage drop and complex current through it. The scalar product is known to be commutative and distributive; therefore we will have: $V\circ I = I\circ V$ and $V\circ(I+I') = V\circ I + V\circ I'$, and so:

$$(V_1+\ldots+V_m)\circ(I_1+\ldots+I_n) = V_1\circ I_1 + \ldots + V_1\circ I_n + \ldots + V_m\circ I_1 + \ldots + V_m\circ I_n. \qquad (4)$$

We do not wish to insist on this here, because it will be treated in a better way in the next section. [Note: If r is real we also have: $(rV)\circ I = V\circ(rI) = r\,V\circ I$.]

§4. COMPLEX (OR VECTOR) POWER.

Let $V=|V|\angle\alpha$ and $I=|I|\angle\beta$ be the complex voltage drop and current through an element of a network in the sinusoidal state. Of the four products: VI, $\bar{V}I$, $V\bar{I}$, $\bar{V}\cdot\bar{I}$, that can be formed with V and I and their conjugates, the only really useful ones are the second and third. Thus, e.g., the complex number given by the product:

$$P = \bar{V}\cdot I = |V|\angle{-\alpha}\cdot|I|\angle\beta = |V|\cdot|I|\angle{\beta-\alpha} = |V|\cdot|I|\cdot\cos(\beta-\alpha) + i\,|V|\cdot|I|\cdot\sin(\beta-\alpha), \qquad (1)$$

has the property that its real part is precisely the mean power taken by the element; and the same is true of the product $V\bar{I}$.

Thus we can expect that these rather artificially constructed complex numbers may be somewhat useful. Since $\bar{V}I$ is the product used more in the literature for the matters to be developed in this section, we will also choose it here.

Accordingly, we will define $P = \bar{V}I$ to be the complex power (also called the vector power) absorbed by an element with the complex voltage drop V and the complex current I through it (when V and I are both referred to the same reference direction); and if $E = -V$, $\bar{E}I$ is defined as the complex power supplied by the element. The modulus $|P| = |V|\cdot|I|$ of the complex power is commonly known as the apparent power, or (as a name) the volt-ampères capacity (abbreviated VA. or va.), of the element; the kilovolt-ampères capacity (abbreviated KVA or kva) is also used (as a name). We have already mentioned above that the real part $W = \mathcal{R}P = \mathcal{R}(\bar{V}I) = |V|\cdot|I| \cos\vartheta$ (where we have put: $\vartheta = \beta - \alpha = \operatorname{ang} P = \operatorname{ang} I - \operatorname{ang} V = -\varphi$) of the complex power is the mean power (also called the active power) taken by the element. Likewise, the imaginary part $W'(\text{say}) = \mathcal{I}P = \mathcal{I}(\bar{V}I) = |V|\cdot|I| \sin\vartheta$ of the complex power is called the reactive power (and also the wattless power) taken by the element. We then have: $P = W + iW'$. (1')

In the MKSC-system of units (cf. Ch. I, §8) the unit of voltage is the volt, and the unit of current is the ampère; hence the unit of power, mean power, apparent power, and reactive power, should be the volt×ampère = watt. However, the watt is usually reserved for power and mean (or active) power, and apparent power is usually given in volt-ampères, which (although the same as the watt) is used even if only to indicate (without an explicit statement to the effect) that one is referring to apparent power and not to mean power. Likewise, reactive power is usually given in reactive volt-ampères (abbreviated var) even if only to indicate that one is referring to reactive power (but reactive watt = rw. would do as well). Complex (or vector) power may be (and is) conventionally measured in units of power (watt), or in units of apparent power (volt-ampère); but rather unnecessarily a vector watt and a vector volt-ampère are commonly used in the literature. (Larger units frequently used are: KV=kv=kilovolt for voltages, kw=KW=kilowatt and MW = megawatt = 10^6 watts for power and mean power (and complex power), kvar=KVAR=reactive kilovolt-ampères for reactive power, and KVA=kva=kilovolt-ampères for apparent and complex power; we recall that kilo = 1000. For small powers, the mw.=milliwatt = thousandth of a watt is frequently used. Concerning currents, the ma.=milliampère=thousandth of an ampère is used for small currents.)

Concerning mean power ($W = V \circ I = \mathcal{R}\bar{V}I$), eq. (4) of §3 may be proved easily as follows:

$$
\begin{aligned}
(V_1+\dots+V_m)\circ(I_1+\dots+I_n) &= \mathcal{R}[(\bar{V}_1+\dots+\bar{V}_m)(I_1+\dots+I_n)] \\
&= \mathcal{R}[\bar{V}_1I_1+\dots+\bar{V}_1I_n+\dots+\bar{V}_mI_1+\dots+\bar{V}_mI_n] \\
&= \mathcal{R}\bar{V}_1I_1+\dots+\mathcal{R}\bar{V}_1I_n+\dots+\mathcal{R}\bar{V}_mI_1+\dots+\mathcal{R}\bar{V}_mI_n \\
&= V_1\circ I_1+\dots+V_1\circ I_n+\dots+V_m\circ I_1+\dots+V_m\circ I_n\,. \qquad (2)
\end{aligned}
$$

In order to obtain an interpretation of the various concepts given above, let us again consider the sinusoidal voltage drop $V^*(t) = A\sin(\omega t+\alpha)$ and the sinusoidal current $I^*(t) = B\sin(\omega t+\beta)$ referred to the same reference direction through a given two-terminal element. The (instantaneous) power absorbed by the element is (see eq. 4º, §2):

$$
\begin{aligned}
W(t) &= V^*(t)I^*(t) = A\sin(\omega t+\alpha)\,B\sin(\omega t+\beta) \\
&= \frac{AB}{2}\cos(\alpha-\beta) - \frac{AB}{2}\cos(2\omega t+\alpha+\beta)\,. \qquad (3)
\end{aligned}
$$

From this we see that the apparent power of the element, namely: $|P| = |AB/2| = |A|/\sqrt{2}\cdot|B|/\sqrt{2} = |V|\cdot|I|$, is precisely the amplitude of the sinusoidal part of the (instantaneous) power absorbed by the element, which is the maximum deviation of the instantaneous power $W(t)$ from its mean value $\overline{W(t)} = (AB/2)\cos(\alpha-\beta)$. It (the apparent power) is then also the maximum rate at which the element can absorb energy in excess of (over) the mean rate at which it can do so. Of course, it is possible to say all this, because the terms in (3) are determined uniquely on account of their linear independence (cf. Ch. IV, §7).

Now let us express $I^*(t)$ as the sum of an in-phase component $I^*_=$ and a quadrature component $I^*_\perp$ respect to $V^*(t)$ as follows (cf. Ch. XI, §5, Prob. 8, p. 289):

$$
\begin{aligned}
I^*(t) &= B\sin(\omega t+\beta) = B\sin(\omega t+\alpha+\beta-\alpha) = \\
&= I^*_=(t) + I^*_\perp(t) = B\cos(\beta-\alpha)\sin(\omega t+\alpha) + B\sin(\beta-\alpha)\cos(\omega t+\alpha)\,. \qquad (4)
\end{aligned}
$$

Then we will also have for the (instantaneous) power absorbed by the element:

$$
\begin{aligned}
W(t) = V^*(t)\,I^*(t) &= AB[\cos(\beta-\alpha)\sin^2(\omega t+\alpha) + \sin(\beta-\alpha)\sin(\omega t+\alpha)\cos(\omega t+\alpha \\
&= \frac{AB}{2}\cos(\beta-\alpha)[1-\cos 2(\omega t+\alpha)] + \frac{AB}{2}\sin(\beta-\alpha)\sin 2(\omega t+\alpha)\,. \qquad (5)
\end{aligned}
$$

The first term in this expression (5) is the power absorbed by the element due to the in-phase component of the current. It never changes sign and thus represents (algebraically speaking) power absorbed irreversibly by the element; and the mean rate at which it absorbs energy is the mean power, the maximum rate being twice the mean power. The second term in (5) is the power absorbed by the *element due to the quadrature component of the current. It represents a to-and-fro-flow of energy to the element (with no net transfer), the energy being stored in the (electromagnetic field associated with the) element and then returned periodically.* (When P is purely imaginary, the mean power $W=0$, and then all the power is of this nature.) *The coefficient $(AB/2)\sin(\beta-\alpha)$ of this term is precisely the reactive power taken by the element; and its amplitude, i.e., the absolute value of the reactive power, is equal to the maximum rate at which energy can be stored in the (electromagnetic field of the) element.*

Problem 1. *By putting $I = I_= + I_\perp$ (cf. Ch. XI, 5, Prob. 8), where $I_=$ is parallel to $\underline{V}$ and $I_\perp$ is at a right angle to $\underline{V}$, show that:*

$$P = W + iW' = \overline{V}I = \overline{V}(I_= + I_\perp) = \overline{V}I_= + \overline{V}I_\perp = \pm|V|\cdot|I_=| \pm i|V|\cdot|I_\perp| . \qquad (6)$$

Thus the active power taken by the element is $W = \pm|V|\cdot|I_=|$ and the reactive power is $W' = \pm|V|\cdot|I_\perp|$. From this we again see that the active power is due only to $I_=$, which is then (also) *called the* *active component* *of the complex current $\underline{I}$; and the reactive power is due only to $I_\perp$, which is then called the* *reactive component* *of $\underline{I}$, also. We may add that* (sometimes) *the reactive power $W' = \pm|V|\cdot|I_\perp|$ is also called* *quadrature power.*

Problem 2. *Show that if $P = |P|\underline{/\vartheta}$, then $W = |P|\cos\vartheta$, $W' = |P|\sin\vartheta$, $|P| = +\sqrt{W^2 + (W')^2}$. Then show that: $W' = W\tan\vartheta = \pm\sqrt{pf.^{-2} - 1}\cdot W$, where the sign to be taken is that of $\vartheta = \operatorname{ang} P$.*

Problem 3. *Show that: $|P| = |W/pf.| = |W'/\sqrt{1-pf.^2}|$. The quantity $\sqrt{1-pf.^2}$ is sometimes called the* *reactive power factor.*

Problem 4. *Show that: $P = (W/pf.)\,\underline{/\pm\cos^{-1}pf.} = \sqrt{W^2 + W'^2}\,\underline{/\pm\tan^{-1}(W'/W)}$.*

Problem 5. *If $V = a + a'i$ and $I = b + b'i$, by performing the product: $P = \overline{V}I = (a - a'i)(b + b'i)$, show that we have for the active and reactive powers, respectively: $W = ab + a'b'$, and $W' = ab' - a'b$.*

Problem 6. *If $\underline{V}$ and $\underline{I}$ are the complex voltage drop and current through a passive magnetically isolated box of impedance $\underline{Z}$ and admittance $Y = 1/Z$, show that the complex power absorbed by the box is:*

$$P = \overline{V}\cdot I = \overline{Z}\,|I|^2 = Y\,|V|^2 , \qquad (7)$$

so that: $$|P| = |V|\cdot|I| = |Z|\cdot|I|^2 = |Y|\cdot|V|^2,$$

and: $$\vartheta = \operatorname{ang} P = \operatorname{ang} I - \operatorname{ang} V = \operatorname{ang}\overline{Z} = -\operatorname{ang} Z = \operatorname{ang} Y .$$

Hence, if $\underline{R}$ and $\underline{X}$ are the equivalent resistance and reactance, and $\underline{G}$ and $\underline{B}$ are the equivalent conductance and susceptance, of the box, show that we have for the active and reactive power absorbed by the box:

$$\begin{aligned} W &= R\,|I|^2 = G\,|V|^2 \\ W' &= -X\,|I|^2 = B\,|V|^2. \end{aligned} \qquad (8)$$

Show also that: $\vartheta = \operatorname{ang} P = \tan^{-1}(-X/R) = \tan^{-1}(B/G)$,

and: $pf. = \cos\vartheta = \cos(\varphi = -\vartheta) = R/|Z| = G/|Y|$;

but notice that: $GR - BX = 1$ & $GX + BR = 0$ (so that perhaps $R \neq 1/G$, etc. (See Ch. VIII, §1, Prob. 2)*).*

<u>Problem 7</u>. Show that the complex power P *taken by* n *two-terminal (active or passive) boxes connected in series is equal to the sum of the complex powers* P_k *taken by the boxes* $k(=1,2,\ldots,n)$. *(<u>Hint</u>:* $P=\overline{V}I=(\overline{V}_1+\ldots+\overline{V}_n)I=\overline{V}_1I+\ldots+\overline{V}_nI=P_1+\ldots+P_n$.*)*

<u>Problem 8</u>. Show that the complex power P *taken by a system of* n *(active or passive) boxes connected in parallel is equal to the sum of the complex powers* P_k *taken by the boxes* $k(=1,2,\ldots,n)$. *(<u>Hint</u>:* $P=\overline{V}(I_1+\ldots+I_n)=\overline{V}I_1+\ldots+\overline{V}I_n=P_1+\ldots+P_n$.*)*

<u>Note 1</u>: As a consequence of the results of the preceding problems 7 & 8, we infer that the complex power taken by any series-parallel combination of (act. or pas. two-ter.) boxes is equal to the sum of the complex powers taken by the individual boxes of the combination. This result is only a particular case of a general principle to be presented in the next section, however.

<u>Problem 9</u>. Show that a magnetically isolated passive box of impedance $Z=R+iX$ *and admittance* $Y=1/Z=G+iB$, *absorbing the complex power* P, *is inductive (i.e.,* $X>0$ *and* $B<0$; *cf. Ch. VIII, §1, p. 217) if and only if* $W'=\mathcal{I}P<0$ *and* $\vartheta=\text{ang}\,P=\text{ang}\,I-\text{ang}\,V=\text{ang}\,Y=-\text{ang}\,Z<0$. *Also show that the box is capacitive* ($B>0$ & $X<0$) *if and only if* $W'=\mathcal{I}P>0$ *and* $\vartheta=\text{ang}\,P=\text{ang}\,I-\text{ang}\,V=\text{ang}\,Y=-\text{ang}\,Z>0$. *Thus, if a magnetically isolated passive box is inductive, the current lags the voltage drop through the box (in a given reference direction), and this is (conventionally) indicated by saying that the box has a <u>lagging power factor</u>; and if the box is capacitive, the current leads the voltage drop, and we say that the box has a <u>leading power factor</u>. To indicate that a power factor is leading* ($\text{ang}\,I>\text{ang}\,V$), *or lagging* ($\text{ang}\,I<\text{ang}\,V$), *we conventionally use a positive, or negative, sign (respectively) as a subscript in the power factor (thus:* $pf._+$ *and* $pf._-$). *Of course, these signs used as subscripts have nothing to do with the sign of the power factor as a quantity, and in fact, both* $pf._+$ *and* $pf._-$ *are positive if* $-90°<\vartheta<+90°$, *whether or not* $\vartheta>0$, *and this shall happen when the equivalent resistance* $R>0$, *in which case the absorbed mean power is positive, so that the box is in effect a sink of energy on the average; and both* $pf._+$ *and* $pf._-$ *are negative when* $|\vartheta|>90°$ *(assuming* $|\vartheta|\leq 180°$), *whether or not* $\vartheta>0$, *and this shall happen when the absorbed mean power is negative (as if the equivalent resistence of the box were negative), i.e., when the supplied mean power is positive, meaning that the box is in effect a source of energy on the average.*

Note 2: Since complex powers are complex numbers, they can be represented graphically as vectors in a Gauss-Argand diagram. The complex (or vector) power diagram of a network should exhibit all the complex powers taken by its individual elements. But, since the vectors of this diagram do not belong to sine functions or rotating vectors, they should not be interpreted as such; and the coordinate axes are essential in these diagrams, since they can not be omitted nor placed arbitrarily. In complex power diagrams, operations with complex powers can be performed graphically.

Problem 10. Show that for a resistance, $P=W+iW'=R|I|^2 \therefore W=R|I|^2=\frac{|V|^2}{R}$ & $W'=0$; for a condenser: $P=W+iW'=i|I|^2/\omega C=i\omega C|V|^2 \therefore W=0$ & $W'=\omega C|V|^2=|I|^2/\omega C$; and for a magnetically isolated coil: $P=W+iW'=-i\omega L|I|^2=|V|^2/i\omega L$ $\therefore W=0$ & $W'=-\omega L|I|^2=-|V|^2/\omega L$. The volt-ampere capacity of a resistance is: $R|I|^2=|V|^2/R$; that of a condenser is: $|P|=\omega C|V|^2=|I|^2/\omega C$; and that of a magnetically isolated coil is: $|P|=\omega L|I|^2=|V|^2/\omega L$. The complex powers absorbed by a resistance, a condenser and a magnetically isolated coil are shown in Fig. 1.

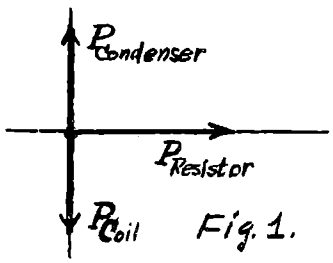


Fig. 1.

Example 1. Consider two loads (=impedance boxes), one taking 50 kw. at a lagging power factor of 0.8, and the other taking 30 kw. at a leading power factor of 0.7, supplied by the same source. The problem is to find the total kva. (=apparent power) and the power factor by which it is supplied by the source.

We know that the total complex power $\underline{P}$ supplied by the source is equal to the sum of the complex powers $P_1=50+iW'_1$ and $P_2=50+iW'_2$ (kw.) absorbed by the loads (whether they are connected in series or in parallel). Now (cf. Prob. 2, above): $W'_1=-\sqrt{0.8^{-2}-1}\times 50=-37.5$ (kvar.) is the reactive power taken by the first load, and: $W'_2=\sqrt{0.7^{-2}-1}\times 30=30.6$ (kvar.) is the reactive power taken by the second load. Hence: $P=P_1+P_2=(50-37.5i)+(30+30.6i)=80-6.9i=80.3\,\underline{/-4^\circ 56'}$. Hence the total kva. is: $|P|=80.3$, and the power factor is: $\cos(-4^\circ 56')=0.996$ (i.e., a lagging power factor of $0.996=99.6\%$).

Of course, P_1 and P_2 can be found directly from the mean powers and power factors (cf. prob. 4, above). Thus: $P_1=(50/0.8)\underline{/-\cos^{-1}0.8}=62.5\,\underline{/-36^\circ 52'}$, and $P_2=(30/0.7)\underline{/+\cos^{-1}0.7}=42.86\,\underline{/45^\circ 34'}$. Thence $P=62.5\underline{/-36^\circ 52'}+42.86\underline{/45^\circ 34'}=80.3\underline{/-4^\circ 56'}$ (as shown in Fig. 2).

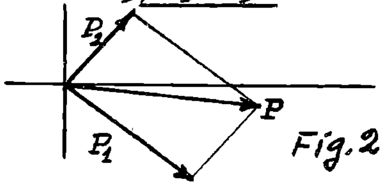


Fig. 2.

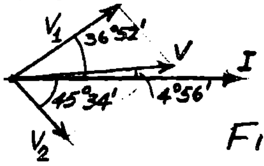


Fig. 3.

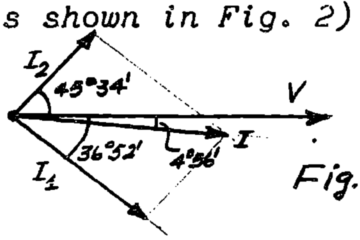


Fig. 4.

If the loads in this example are connected in series, they will have a common current through them and the (ordinary) vector diagram will be of the form shown in Fig. 3 (with the current taken as reference). The total voltage drop will be divided between the two loads in such a way that each load absorbs the corresponding power at the corresponding power factor, with the same current through them.

On the other hand, if the loads are connected in parallel, they will have the same voltage drop across them, and the total current will divide between them in such a way that each load absorbs the corresponding power at the corresponding power factor (as shown in Fig. 4, with the common voltage drop through them taken as reference).

Problem 11. Three loads are connected in parallel across a sinusoidal voltage source with an effective voltage of 220 volts, at 50∿. The first takes 100 kw at a lagging power factor of 0.85 (i.e., pf. = $0.85\_$); the second takes an active power of 75 kw. and a reactive power of 25 kvar.; and the third takes a current of 100 amp. in phase with the voltage of the source. Find the total complex power supplied to these loads, the total kva. (apparent power), the total effective current, the resulting power factor (of the combination) the total complex current, the total active and reactive powers, and the total sinusoidal current. (Take the angle of the voltage = 0.)

Example 2. In a certain factory there is an induction motor (= an inductive two-terminal box, for our purposes) taking 100 kw. at a pf. = $0.7\_$. Another motor, to take 10 kw., is to be added to the factory for some other purpose, and it is desired to take the opportunity to increase the resulting power factor to unity. For this, a synchronous motor to be operated (over-excited) at a leading power factor (and to be considered as a capacitive two-terminal box for our purposes) is chosen to take the new 10 kw. load. The problem is to find the kva. capacity which the synchronous motor must have and the (leading) power factor at which it must be operated (by over-excitation).

Let $P_1 = W_1 + i\,W_1'$ denote the complex power taken by the induction motor, and $P_2 = W_2 + i\,W_2'$ that to be taken by the synchronous motor. Then we have: $W_1 = 100$ kw. and $W_1' = -\sqrt{0.7^{-2} - 1} \times 100 = -102$ kvar. Also: $W_2 = 10$ kw., and we must have $\mathscr{I}P = \mathscr{I}(P_1 + P_2) = W_1' + W_2' = 0$, in order that $\operatorname{ang} P = 0$ and so pf. $= \cos 0 = 1$. Thus we must have: $W_2' = -W_1' =$ 102 kvar., so that $P_2 = 10 + 102\,i = 102.5\,\underline{/84^\circ 24'}$. Therefore the kva. capacity of the synchronous motor must be 102.5 kva. and it must be operated at a leading power factor of $\cos 84^\circ 24' = 0.097 = 9.7\%$.

<u>Example 3</u>. Let us assume that in the preceding problem only a resulting power factor of 0.9_- *were asked for. Then we would proceed as follows. Let* $\theta = -\cos^{-1} 0.9 = -25°50'$. *Then:* $(W_1'+W_2')/(W_1+W_2) = (-102+W_2')/(100+10) = \tan(-25°50') = -0.484$. *Hence:* $W_2' = -0.484 \times 110 + 102 = 48.7$ *kvar. Therefore:* $P_2 = 10 + 48.7\,i = 49.8 \angle 78°25'$, *and so the kva. capacity of the synchronous motor would only have to be* $|P_2| = 49.8$ *(kva.) and it would only have to be operated at a (leading) pf.* $= \cos 78°25' = 0.202_+$.

<u>Problem 12</u>. Show that if two loads absorb complex powers $P_1 = p_1 \angle \vartheta_1 = W_1 + i\,W_1'$ *and* $P_2 = W_2 + i\,W_2' = p_2 \angle \vartheta_2$, *respectively, and the total power supplied to them is* $P = W + i\,W' = p \angle \vartheta$, *then we have:* $W = W_1 + W_2$, $W' = W_1' + W_2'$, $p^2 = (W_1 + W_2)^2 + (W_1' + W_2')^2$, $\tan\vartheta = (W_1' + W_2')/(W_1 + W_2)$, $p_1/\sin(\vartheta_2 - \vartheta) = p_2/\sin(\vartheta - \vartheta_1) = p/\sin(\vartheta_2 - \vartheta_1)$. *(<u>Hint</u>: Draw the complex power diagram.)*

<u>Problem 13</u>. If an induction motor takes 50 kw. at a $pf._1 = 0.8_-$ *(lagging), and a synchronous motor connected to the same source is operated at a* $pf._2 = 0.4_+$ *(leading), what should the kva. capacity of the latter be in order that the resulting pf. of the combination be* 0.9_+ *(leading)? What should its kva. capacity be in order that the kva. (apparent power) of the combination be a minimum? If the synchronous motor were operated at 20 kva. what should its power factor be in order that the kva. of the combination be the same as that of the induction motor alone, namely, 50/0.8 = 62.5 kva.?*

<u>Problem 14</u>. From Waldo V. <u>Lyon's</u> book: <u>Problems in Electrical Engineering, Alternating Currents</u> (McGraw-Hill Book Co., 1931, 2nd edition) do the following problems: Of chapter I, problems number 160, 161, 198, 244, 316, 318, 319, 320, 326, 333, 338, 342, 344, 350, 351, 352, 353; and of chapter II, problems number 209, 210, 211, 212, 213, 216, 217, 218. [<u>Note</u>: In Lyon's book, voltages (and potentials) and currents mean effective values of voltages and currents, respectively. A "constant alternating e.m.f." means a sinusoidal emf. "with a constant amplitude" (?); and "constant-potential 60-cycle (per second) mains" means a voltage source with a sinusoidal value (=emf.) and a 60 cycles per second frequency. Motors are to be taken as impedance boxes (=loads). (Power) stations and generators are to be taken as sources. An "impedance coil" means a practical coil with resistance and inductance, so that in our terminology it means a resistor in series with an idealized coil. The "complex expression of a current or voltage" means the complex current or voltage. A "condenser with a pf. $\neq 0$*" means a leaky condenser (with some conductance), or a resistor in series with a condenser.]*

§5. THE CONSERVATION OF COMPLEX POWER PRINCIPLE.

Consider an arbitrary sinusoidal current network of n_e two-terminal (active or passive) elements of any kind whatsoever (to be considered as units) oriented and numbered consecutively from 1 to n_e in any way. Let V_k and I_k denote the complex (terminal) voltage drop and current through the element k in the assigned reference directions. Assume the elements to be connected into n_n nodes in n_c components (=separate parts). The number of independent nodes will be $n_n - n_c = n'_n$, and the number of independent meshes will be $n_e - n'_n = n_m$. Omitting exactly one node in each component, let us number the rest of the nodes consecutively from 1 to n'_n. Also, choosing a complete independent set of n_m meshes in the usual way, let us arbitrarily orient and number them consecutively from 1 to n_m.

The complete set of equations expressing Kirchhoff's laws for the independent nodes n and meshes m of the network will then be:

$$\sum_{k=1}^{n_e}(k,n)I_k = 0 \quad (n=1,2,\dots,n'_n), \qquad \sum_{k=1}^{n_e}[k,m]\,V_k = 0 \quad (m=1,2,\dots,n_m), \tag{1}$$

where (k,n) is the incidence number of the whole unit k (considered as an element) with the node n, and $[k,m]$ is the incidence number of the element k with the mesh m. The equations relating the complex currents and voltages through the elements will be irrelevant here.

Now let us express the complex currents I_k $(k=1,2,\dots,n_e)$ through the units in terms of the complex circulating currents J_m $(m=1,2,\dots,n_m)$ round the meshes, as follows (cf. Ch. VI, Intro.):

$$I_k = \sum_{m=1}^{n_m}[k,m]\,J_m \qquad (k=1,2,\dots,n_e). \tag{2}$$

We know that the first of eqs. (1) are then satisfied identically. Moreover, by taking the complex conjugates in the other eqs. (2), we have:

$$\sum_{k=1}^{n_e}[k,m]\,\overline{V}_k = 0 \qquad (m=1,2,\dots n_m). \tag{3}$$

Hence (using eqs. 2):

$$\sum_{k=1}^{n_e}P_k = \sum_{k=1}^{n_e}\overline{V}_k I_k = \sum_{k=1}^{n_e}\overline{V}_k\sum_{m=1}^{n_m}[k,m]J_m = \sum_{m=1}^{n_m}\left(\sum_{k=1}^{n_e}[k,m]\,\overline{V}_k\right)J_m = \sum_{m=1}^{n_m}0\cdot J_m = 0, \tag{4}$$

by eqs. (3). That is, the algebraic sum of the complex powers absorbed (or supplied, if all the terms of eq. 4 are multiplied by -1) by the totality of the elements of the network is zero; and this is precisely the principle of the conservation of complex power.

Problem 1. Show that the above principle holds for each component a $(=1,2,\dots,n_c)$ separately, i.e., $\sum_{k\in a}P_k = 0$ $(a=1,2,\dots,n_c)$, (5) where $\sum_{k\in a}$ denotes a summation over the elements of component a only.

Problem 2. Instead of using eqs. (2), prove the principle of the conservation of complex power by expressing the complex voltage drops V_k $(k=1,2,..,n_e)$ in terms of the complex nodal potentials U_n $(n=1,2,...,n'_n)$ as follows (cf. Ch. VII, Intro.):

$$V_k = \sum_{n=1}^{n'_n}(k,n)U_n \qquad (k=1,2,...,n_e). \tag{6}$$

Problem 3. Prove that principles of conservation of each of the products: $P'_k = V_k\bar{I}_k$, $P''_k = V_k I_k$, $P'''_k = \bar{V}_k\bar{I}_k$ also hold, i.e.,

$$\sum_{k=1}^{n_e} P'_k = 0\ , \qquad \sum_{k=1}^{n_e} P''_k = 0\ , \qquad \sum_{k=1}^{n_e} P'''_k = 0\ . \tag{7}$$

(The summations may be limted to the elements of a given component.)

Problem 4. Prove the principle of conservation of mean (or active) power, namely:

$$\sum_{k=1}^{n_e} W_k = 0, \quad \text{or} \quad \sum_{k\in a} W_k = 0 \quad (a=1,2,...,n_c). \tag{8}$$

This, however, is an easy consequence of the principle of conservation of energy of classical physics; because the latter implies the conservation of instantaneous power and so, by integration, it implies the conservation of mean power. (Hint: Add $W_k = (P_k + P'_k)/2$.)

Problem 5. Prove the principle of conservation of reactive (or wattless) power, namely:

$$\sum_{k=1}^{n_e} W'_k = 0, \quad \text{or} \quad \sum_{k\in a} W'_k = 0 \quad (a=1,...,n_c). \tag{9}$$

(Hint: Add $W'_k = (P_k - P'_k)/2 = (\bar{V}_k I_k - V_k\bar{I}_k)/2$; cf. Prob. 3, above.)

The above principle may be put into various alternative forms. In the first place, let us suppose that all the elements of the network (or of a given component) are grouped in two classes: K'= the class of active elements, and K''= the class of passive elements (of the network or of the given component). Then the sum of the complex powers absorbed by the elements of the network (or of the given component) may be split in two: $\sum P_k = \sum' P_k + \sum'' P_k$ (= 0, by the principle of conservation of complex power), where the first sum is taken over the class K' of active elements, and the second is taken over the class K'' of passive elements. Hence: $\sum'(-P_k) = \sum'' P_k$. Now, for k in K', $(-P_k) = -\bar{V}_k I_k = \bar{E}_k I_k$ is the complex power supplied by the active element k (E_k being the complex voltage rise through it, in the given reference direction). Consequently, we have that the sum of the complex powers supplied by the active elements of the network (or of the given component) is equal to the sum of the complex powers absorbed by the passive elements of the network (or of the given component). In particular, since resistors absorb no reactive power, *and condensers and magnetically isolated (ideal) coils absorb no active power (cf. Prob. 10, §4), we infer that all the active power supplied by the (current and voltage) sources of a network (or of a given component) is absorbed entirely by the resistors and magnetically coupled coils of the network (or component), and all the reactive power supplied by the sources is absorbed entirely by the condensers and coils of the network (or component). In the next section we will show that the complete (or any closed) system of (ideal) coils of a network absorb no (=zero) active power. Hence they only pass the active power around from one to the other, as if it were so much water; and, in the end, all the active power supplied by the sources of a network is absorbed exclusively by its resistors; but this is not true for a single component in any network. This means that the condensers and the complete system of coils of a network absorb no energy on the average; all they do is store the energy they receive at times and then return it.*

Another important form of the principle of conservation of complex power is obtained as follows. Suppose that the network can be considered as formed by two parts A & B connected together by various (impedanceless) conductors as shown in Fig. 1. Of course, these parts A & B may also be coupled together magnetically and each may have various components. Then we have: $\sum_A P_k + \sum_B P_k = 0$, *where the first sum is taken over all the elements in part A, and the second is taken over all the elements in part B. Consequently:*

Fig. 1.

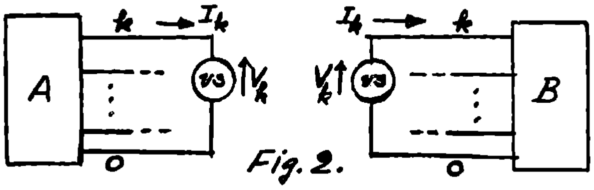


Fig. 2.

$\sum_A (-P_k) = \sum_B P_k$ *; that is, the (algebraic) sum of all the complex powers supplied by the elements in part A (=the net complex power supplied by the part A) is equal to the (algebraic) sum of the complex powers absorbed by the elements in part B (=the net complex power absorbed by the part B). Considering that the (net) complex power supplied (or absorbed) by a component of a network is zero (cf. Prob. 1), the above well-defined quantity may be considered as the flow of complex power out from part A into part B through the conductors connecting them together.*

An expression for the complex power flowing from a part A of a non-singular network to a part $\underline{B}$ through $\underline{N}$ conductors (without impedance) connecting them together can be obtained easily in terms of the complex currents through, and the complex voltages between, these conductors. Without loss of generality we may assume that the parts $\underline{A}$ & $\underline{B}$ form a single component. Let one of the conductors joining $\underline{A}$ to $\underline{B}$ be taken as a base node $\underline{O}$, and let the others be numbered consecutively from 1 to $N-1$. Let I_k $(k=1,2,\ldots,N-1)$ denote the complex current through the conductor $\underline{k}$ (from $\underline{A}$ to $\underline{B}$) and let V_k denote the complex voltage drop from conductor $\underline{k}$ to the base node $\underline{O}$. Now if (we imagine that) all the conductors joining $\underline{A}$ to $\underline{B}$ are cut in between $\underline{A}$ and $\underline{B}$, and at the same time, for each $k=1,2,\ldots,N-1$, a pair of voltage sources of complex emfs. V_k are inserted (one to the left and the other to the right of the cut) from the two halves of the base node $\underline{O}$ to the two halves of the conductor $\underline{k}$ (as shown in Fig. 2) while the magnetic couplings between the parts $\underline{A}$ & $\underline{B}$ (if any) are not disturbed, then (assuming that the network remains non-singular) it is easily proved by the guess and check method (with the help of the existence and uniqueness theorems) that the voltages and currents in parts $\underline{A}$ and $\underline{B}$ are unaffected (cf. Ch. IX). As a consequence, the complex powers supplied by the elements of the part $\underline{A}$, and those absorbed by the elements of the part $\underline{B}$, remain unaltered; and hence so do the net complex powers supplied and absorbed by parts $\underline{A}$ & $\underline{B}$. But by the principle of conservation of complex power applied to the left and right portions of the divided network shown in Fig. 2, respectively, we infer that the complex power supplied by part $\underline{A}$ is equal to the complex power absorbed by the sources connected (on the left of the cut) to part $\underline{A}$, and that the complex power absorbed by part $\underline{B}$ is the same as the complex power supplied by the sources connected (on the right of the cut) to part $\underline{B}$; both of which are equal to:

$$P_{AB}\ (\text{say}) = \sum_{k=1}^{N-1} \bar{V}_k I_k \,, \tag{10}$$

which is the expression sought for the flow of complex power from part $\underline{A}$ to part $\underline{B}$ through the conductors joining them.

It is easy to see that the quantity P_{AB} given by eq. (10) does not depend on the particular conductor $\underline{O}$ chosen as the reference conductor. For, in the first place, by Ch. II, §3, Cor. 1, we know that the sum of (the instantaneous, and so the sinusoidal, and so) the complex currents through the $\underline{N}$ conductors (from $\underline{A}$ to $\underline{B}$) is zero;

that is $\sum_{k=0}^{N-1} I_k = 0$; and in the second place, if V_{kl} denotes the complex voltage drop from the conductor $\underline{k}$ $(=0,1,\dots,N-1)$ to any other conductor $\underline{l}$, we have $V_k = V_{kl} + V_l$ and so:

$$\sum_{k=0}^{N-1} \bar{V}_{kl} I_k = \sum_{k=0}^{N-1} (\bar{V}_k - \bar{V}_l) I_k = \sum_{k=0}^{N-1} \bar{V}_k I_k - \bar{V}_l \cdot \sum_{k=0}^{N-1} I_k = \sum_{k=1}^{N-1} \bar{V}_k I_k .$$

By taking real parts in eq. (10), we have for the active (=mean) power W_{AB} (say) flowing from part $\underline{A}$ to part $\underline{B}$:

$$W_{AB} = \mathcal{R} P_{AB} = \sum_{k=1}^{N-1} \mathcal{R}(\bar{V}_k I_k) = \sum_{k=1}^{N-1} V_k \circ I_k . \qquad (11)$$

If the current elements of $N-1$ wattmeters are inserted in the $N-1$ conductors $1, 2, \dots, N-1$ and the corresponding voltage elements of the wattmeters are connected across from the conductors $1, 2, \dots, N-1$ to the reference conductor $\underline{0}$, the wattmeter inserted in the conductor $\underline{k}$ will measure $V_k \bullet I_k$; and eq. (11) tells us that the algebraic sum of the readings of the $N-1$ wattmeters is the mean power flowing from part $\underline{A}$ to part $\underline{B}$. This is called the (N-1)-wattmeter method of measuring the active (=mean) power absorbed by the part $\underline{B}$. Of course, the $N-1$ wattmeters could be incorporated into a single multi-wattmeter which would effect this sum internally and give the result in a single reading on a single scale.

§6. THE DISTRIBUTION OF COMPLEX POWER IN A SYSTEM OF BOXES.

Consider a (closed) system of $\underline{N}$ passive two-terminal boxes in the sinusoidal state, oriented and numbered arbitrarily from 1 to $\underline{N}$, with the self- and mutual-impedances Z_{kl} and admittances Y_{kl} (k & $l = 1,2,\dots,N$) with respect to the assigned orientations (=reference directions). The complex voltage drop V_k across the box $\underline{k}$ is given in terms of the complex (terminal) currents I_k through the boxes $\underline{l}$ $(=1,2,\dots,N)$ in the assigned reference directions by the following equation (cf. Ch. VIII, §2):

$$V_k = \sum_{l=1}^{N} Z_{kl} I_l \qquad (k = 1,2,\dots,N). \qquad (1)$$

The conjugates of these are:

$$\bar{V}_k = \sum_{l=1}^{N} \bar{Z}_{kl} \bar{I}_l \qquad (k = 1,2,\dots,N),$$

and so the complex power absorbed by the box $\underline{k}$ is:

$$P_k = \bar{V}_k I_k = \sum_{l=1}^{N} \bar{Z}_{kl} I_k \bar{I}_l . \qquad (k = 1,2,\dots,N) \qquad (2)$$

In terms of the admittances we have (cf. Ch. VIII, §2):

$$I_k = \sum_{l=1}^{N} Y_{kl} V_l \qquad \therefore \qquad P_k = \bar{V}_k I_k = \sum_{l=1}^{N} Y_{kl} \bar{V}_k V_l , \quad (k = 1,2,\dots,N). \quad (3)$$

The distribution of the complex powers P_k amongst the boxes of the system can be found from eqs. (2), when their impedances Z_{kl} and terminal complex currents I_k are known, and from eqs. (3), when the admittances Y_{kl} and the complex voltage drops V_k across the boxes are known.

The total complex power absorbed by the system of boxes, that is, the sum of the complex powers taken by the boxes of the system, is then:

$$P = \sum_{k=1}^{N} P_k = \sum_{k=1}^{N}\sum_{l=1}^{N} \bar{Z}_{kl} I_k \bar{I}_l = \sum_{k=1}^{N}\sum_{l=1}^{N} Y_{kl} \bar{V}_k V_l . \qquad (4)$$

Since the impedances and admittances are symmetric (cf. Ch. VIII, §2, Prob. 2, p. 224), that is: $Z_{kl}=Z_{lk}$ & $Y_{kl}=Y_{lk}$ (k & $l = 1,2,\ldots,N$), we have:

$$\sum_{k=1}^{N}\sum_{l=1}^{N} \bar{Z}_{kl} I_k \bar{I}_l = \sum_{l=1}^{N}\sum_{k=1}^{N} \bar{Z}_{kl} I_l \bar{I}_k = \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} \bar{Z}_{kl}(I_k \bar{I}_l + \bar{I}_k I_l),$$

and similarly for the expression in terms of Y_{kl} and the V_k. Thus we have for the total complex power absorbed by the system of boxes:

$$\begin{aligned} P = \sum_{k=1}^{N} P_k &= \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} \bar{Z}_{kl}(I_k \bar{I}_l + \bar{I}_k I_l) = \sum_{k=1}^{N}\sum_{l=1}^{N} \bar{Z}_{kl}\, \mathcal{R}(I_k \bar{I}_l) \\ &= \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} Y_{kl}(\bar{V}_k V_l + V_k \bar{V}_l) = \sum_{k=1}^{N}\sum_{l=1}^{N} Y_{kl}\, \mathcal{R}(V_k \bar{V}_l). \end{aligned} \qquad (5)$$

In particular, for a (closed) system of N (ideal) coils in the sinusoidal state at the angular frequency ω, with the self- and mutual-inductances L_{kl} (k & $l = 1,2,\ldots,N$) referred to assigned orientations, we have: $Z_{kl} = i\omega L_{kl} = Z_{lk}$ and $Y_{kl} = \Gamma_{kl}/i\omega = Y_{lk}$. Therefore the total complex power taken by the system of coils is:

$$-i\omega \sum_{k=1}^{N}\sum_{l=1}^{N} L_{kl}\, \mathcal{R}(I_k \bar{I}_l) = \frac{1}{i\omega}\sum_{k=1}^{N}\sum_{l=1}^{N} \Gamma_{kl}\, \mathcal{R}(V_k \bar{V}_l). \qquad (6)$$

These expressions are pure imaginaries, because all the quantities within the double summations are real. Thus the system of (ideal) coils absorbs only reactive power; and the active (=mean) power taken by the whole system of (ideal) coils is zero (as we said in §5).

Problem 1. Consider a (closed) system of N active two-terminal boxes in the sinusoidal state, arbitrarily oriented and numbered from 1 to N. Let Z_{kl} and Y_{kl} be the passified (self- and mutual-) impedances and admittances (resp.) of the boxes, and let E°_k be the open-circuit complex EMF., and I°_k the short-circuit complex current, of the box k (cf. Ch. VIII, §5, pp. 240-1). Show that the complex power absorbed by the box $k(=1,2,\ldots,N)$ is:

$$P_k = \bar{V}_k I_k = \sum_{l=1}^{N} \bar{Z}_{kl} I_k \bar{I}_l - \bar{E}^\circ_k I_k = \sum_{l=1}^{N} Y_{kl} \bar{V}_k V_l - \bar{V}_k I^\circ_k \;, \qquad (7)$$

and hence that the total complex power absorbed by the system is:

$$P = \sum_{k=1}^{N} P_k = \sum_{k=1}^{N}\sum_{l=1}^{N} \bar{Z}_{kl} I_k \bar{I}_l - \sum_{k=1}^{N} \bar{E}^\circ_k I_k = \sum_{k=1}^{N}\sum_{l=1}^{N} Y_{kl} \bar{V}_k V_l - \sum_{k=1}^{N} \bar{V}_k I^\circ_k \;. \qquad (8)$$

Problem 2. By comparing the expressions (4) and (5), show that:

$$\sum_{k=1}^{N}\sum_{l=1}^{N}\bar{Z}_{kl}I_k\bar{I}_l=\sum_{k=1}^{N}\sum_{l=1}^{N}\bar{Z}_{kl}\mathcal{R}(I_k\bar{I}_l) \quad \& \quad \sum_{k=1}^{N}\sum_{l=1}^{N}Y_{kl}\bar{V}_kV_l=\sum_{k=1}^{N}\sum_{l=1}^{N}Y_{kl}\mathcal{R}(\bar{V}_kV_l).$$

In particular, show that for a (closed) system of (ideal) coils the following expressions are always real:

$$\sum_{k=1}^{N}\sum_{l=1}^{N}L_{kl}I_k\bar{I}_l \qquad \text{and} \qquad \sum_{k=1}^{N}\sum_{l=1}^{N}\Gamma_{kl}\bar{V}_kV_l.$$

[This can be checked by showing that they are equal to their conjugates.]

Problem 3. By expressing the (self- and mutual-) impedances $Z_{kl}=R_{kl}+iX_{kl}$ and admittances $Y_{kl}=G_{kl}+iB_{kl}$ in terms of the equivalent resistances $R_{kl}=\mathcal{R}Z_{kl}$ and equivalent reactances $X_{kl}=\mathcal{J}Z_{kl}$, and the equivalent conductances $G_{kl}=\mathcal{R}Y_{kl}$ and susceptances $B_{kl}=\mathcal{J}Y_{kl}$ (resp.), show (1º) that the following quantities are real:

$$\sum_{k=1}^{N}\sum_{l=1}^{N}R_{kl}I_k\bar{I}_l=\sum_{k=1}^{N}\sum_{l=1}^{N}R_{kl}\bar{I}_kI_l\ , \quad \sum_{k=1}^{N}\sum_{l=1}^{N}X_{kl}I_k\bar{I}_l=\sum_{k=1}^{N}\sum_{l=1}^{N}X_{kl}\bar{I}_kI_l$$

and similarly for $\sum\sum G_{kl}V_kV_l$ & $\sum\sum B_{kl}V_kV_l$; and (2º) that the total active and reactive powers absorbed by the system of boxes are:

$$W=\sum_{k=1}^{N}W_k=\sum_{k=1}^{N}\sum_{l=1}^{N}R_{kl}I_k\bar{I}_l=\sum_{k=1}^{N}\sum_{l=1}^{N}G_{kl}\bar{V}_kV_l \tag{9}$$

$$W'=\sum_{k=1}^{N}W'_k=-\sum_{k=1}^{N}\sum_{l=1}^{N}X_{kl}I_k\bar{I}_l=\sum_{k=1}^{N}\sum_{l=1}^{N}B_{kl}\bar{V}_kV_l. \tag{10}$$

{Notice that: $G_{kl}=\mathcal{R}[\operatorname{cof}Z_{lk}/\det(Z_{kl})]$, $B_{kl}=\mathcal{J}[\operatorname{cof}Z_{lk}/\det(Z_{kl})]$.}

Let us again consider eqs. (1) and (2) for the complex voltage drop across, and the complex power absorbed by, the box k. The typical term $Z_{kl}I_l$ of the sum (1) is the contribution of the complex current I_l through the box l to the complex voltage drop across the box k. Therefore, the typical term $\bar{Z}_{kl}I_k\bar{I}_l$ of the sum (2) will then be the contribution of the complex current I_l through the box l to the complex power absorbed by the box k from the network. This complex power is then sent to the thermic and electromagnetic fields associated with the mutual-impedance Z_{kl}; and the negative of this quantity, namely: $-\bar{Z}_{kl}I_k\bar{I}_l$, may be considered as the flow of complex power from those fields to the box k (algebraically speaking), and from there to the network. Eq. (2), after multiplying throughout by -1, would then state that the complex power supplied by the box k to the network is equal to the sum of all the flows of complex power from the thermic and electromagnetic fields associated with all the self- and mutual-impedances of the box k with the other boxes of the system and with itself. Of course, similar remarks can be made with the help of eqs. (3).

When an arbitrary network in the sinusoidal state at the angular frequency ω is considered as a network of n_e elements of the general series type (shown in Fig. 1), for which we have (cf. Ch. V, §1):

$$V_k = R_k I_k + (S_k/i\omega) I_k + \sum_{l=1}^{n_e} i\omega L_{kl} I_l - E_k - D_k \quad (k=1,\ldots,n_e) \qquad (11)$$

the principle of conservation of complex power (eq. 4, §5) gives:

$$\sum_{k=1}^{n_e} \bar{V}_k I_k = \sum_{k=1}^{n_e} (R_k \bar{I}_k + iS_k \bar{I}_k/\omega - \sum_{l=1}^{n_e} i\omega L_{kl} \bar{I}_l - \bar{E}_k - \bar{D}_k) I_k = 0,$$

which may be written as follows:

$$\sum_{k=1}^{n_e} \bar{E}_k I_k + \sum_{k=1}^{n_e} \bar{D}_k I_k = \sum_{k=1}^{n_e} R_k |I_k|^2 + i \sum_{k=1}^{n_e} S_k |I_k|^2/\omega - i \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} \omega L_{kl} I_k \bar{I}_l . \qquad (12)$$

The left-hand member in this equation is the total complex power supplied by all the sources (=the active elements) of the network, while the right-hand member is the total complex power absorbed by all the passive elements of the network; and so we have here another proof of the second form of the principle of conservation of complex power given in §5.

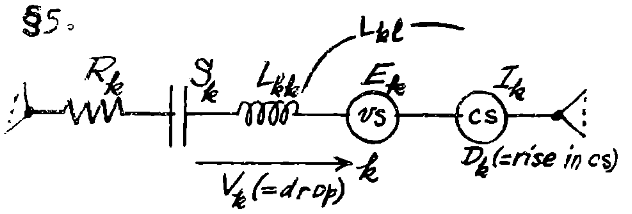


Fig. 1

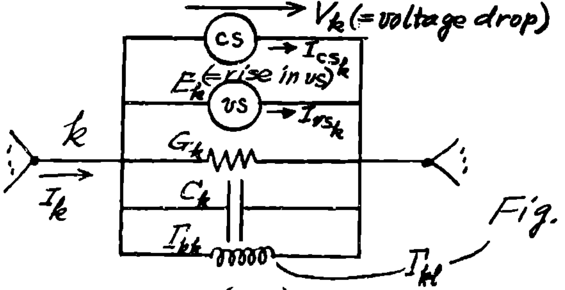


Fig. 2.

The first term in the left member of eq. (12) is the total complex power supplied by the voltage sources of the network, and the second term is that supplied by all the current sources. ¶*The first term in the right member of eq. (12) is the total complex power absorbed by all the resistors of the network, and this is all mean or active power (since it is real), all of which is transformed irreversibly into heat of the thermic field of the network. If I_k^* denotes the sinusoidal current through the element* <u>k</u>*(=1,2,...,n_e), we know that the instantaneous rate at which the resistor* <u>k</u> *transforms the energy it absorbs from the network into heat is $R_k I_k^{*2}$, the mean value of which is $R_k |I_k|^2$. Hence:*

$$\sum_{k=1}^{n_e} R_k |I_k|^2 = \bar{W}_H \text{ (say)} \qquad (13)$$

is the mean value of the total power dissipated into heat by the (resistors of the) network.

The second term in the right member of eq. (12) is the total complex power absorbed by all the condensers of the network, and this is all due to reactive power (since it is a pure imaginary).

If Q_k^* denotes the sinusoidal charge on the condenser of element $\underline{k}$, we know (cf. Ch. I, §7, Prob. 2) that $\frac{d}{dt}(\frac{1}{2}S_kQ_k^{*2})$ is the rate at which the condenser $\underline{k}$ absorbs energy from the network; so that $S_kQ_k^{*2}/2$ is the instantaneous value of the energy stored by the condenser in its electric field. Denoting the complex charge on the condenser $\underline{k}$ by Q_k, the mean value of this stored energy is: $S_k|Q_k|^2/2 = S_k|i\omega Q_k|^2/2\omega^2 = S_k|I_k|^2/2\omega^2$. Therefore we will have:

$$\sum_{k=1}^{n_e} S_k|I_k|^2/\omega = 2\omega\bar{U}_E \text{ (say)} \tag{14}$$

where $\bar{U}_E$ is the mean value of the total energy stored in the condensers of the network in the associated electric field.

The third term in the right member of eq. (12) is the total complex power absorbed by the system of coils of the network, and this is all due to reactive power, since it is a pure imaginary (cf. Prob. 2). Now we know (cf. Ch. I, §7, eq. 5') that the instantaneous rate at which the system of coils absorbs energy from the network to store it in its magnetic field is given by the derivative of the quantity: $\frac{1}{2}\sum_{k=1}^{n_e}\sum_{l=1}^{n_e} L_{kl}I_k^*I_l^* = U_M$ (say). This U_M is then the total instantaneous value of the energy stored in the magnetic field associated with the system of coils, the mean value of which is:

$$\bar{U}_M = \frac{1}{2}\sum_{k=1}^{n_e}\sum_{l=1}^{n_e} L_{kl}|I_k||I_l| \cos(\text{ang } I_k - \text{ang } I_l) = \frac{1}{2}\sum_{k=1}^{n_e}\sum_{l=1}^{n_e} L_{kl}\mathcal{R}(I_k\bar{I}_l) = \frac{1}{2}\sum_{k=1}^{n_e}\sum_{l=1}^{n_e} L_{kl}I_k\bar{I}_l \tag{15}$$

according to Prob. 2. Therefore, the coefficient of i in the third term in the right member of eq. (12) is precisely $-2\omega\bar{U}_M$.

Substituting the above expressions found for the three terms in the second member of eq. (12), we obtain for the total complex power $\underline{P}$ (say) supplied by the sources of the network and absorbed by its passive elements:

$$P = \sum_{k=1}^{n_e}(\bar{E}_k + \bar{D}_k)I_k = \bar{W}_H + 2\, i\omega(\bar{U}_E - \bar{U}_M). \tag{16}$$

The real part of this is the total active or mean power $W = \bar{W}_H$, and the imaginary part is the total reactive power $W' = 2\omega(\bar{U}_E - \bar{U}_M)$. It is to be noticed that the quantities $\bar{W}_H$, $\bar{U}_E$, $\bar{U}_M$, are never negative. In fact, when all the elements have non-vanishing resistance, elastance, and inductance, these quantities are always positive unless all the I_k and Q_k are zero (not considering, of course, negative resistances, elastances, or self-inductances—which would be equivalent to sources, in a sense).

If the network has a single source, connected between the points $\underline{a}$ & $\underline{b}$ of the network, through which there is a complex current $\underline{I}$ and a complex EMF. $\underline{E}$, the total complex power supplied by the source(s) of the network reduces to: $\bar{E}I = \bar{W}_H + 2i\omega(\bar{U}_E - \bar{U}_M)$, where the quantities *in this equation are now those of this network. The so-called* *__driving-point__* *__impedance__* *__Z__* *__and__* *__admittance__* *__Y__* *of the network between the points* *__a__* *&* *__b__* *will then be given by:*

$$Z = E/I = E\bar{I}/|I|^2 = \left[\bar{W}_H + 2\,i\omega(\bar{U}_M - \bar{U}_E)\right]/\,|I|^2 = 1/Y, \tag{17}$$

$$Y = I/E = \bar{E}I/\,|E|^2 = \left[\bar{W}_H + 2i\omega(\bar{U}_E - \bar{U}_M)\right]/\,|E|^2 = 1/Z. \tag{18}$$

It can be observed that the network as viewed from the driving points *__a__* *&* *__b__* *will be inductive or capacitive according to whether* $\bar{U}_M >$ *or* $< \bar{U}_E$, *i.e., according to whether the mean energy stored in the magnetic field associated with the coils of the network is greater or smaller than the mean energy stored in the electric field associated with the condensers of the network, respectively.*

__Problem 4.__ *By expressing eqs. (11) in terms of the impedances* Z_{kl} *of the elements (cf. Ch. V, §1, eqs. 15), namely:*

$$V_k = \sum_{l=1}^{n_e} Z_{kl} I_l - E_k - D_k \qquad (k=1,2,\ldots,n_e), \tag{19}$$

show that the total complex power supplied by the sources is:

$$P = \sum_{k=1}^{n_e}\sum_{l=1}^{n_e} \bar{Z}_{kl}\, I_k \bar{I}_l \,. \tag{20}$$

Then, replacing the complex currents I_k *through the elements* $k(=1,2,\ldots,n_e)$ *in terms of the complex circulating currents* J_m *(cf. Ch. VI) round the meshes* $m(=1,2,\ldots,n_m)$ *of the network, show that:*

$$P = \sum_{m=1}^{n_m}\sum_{n=1}^{n_m} \bar{z}_{mn}\, J_m \bar{J}_n \,, \tag{21}$$

where the z_{mn} *are the mesh impedances (cf. Ch. VI, §2). Finally, by expressing the mesh impedances in terms of the mesh resistances* r_{mn}, *mesh elastances* s_{mn}, *and mesh inductances* l_{mn} *(cf. Ch. VI, §2, Prob. 2), show that:*

$$\bar{W}_H = \sum_{m=1}^{n_m}\sum_{n=1}^{n_m} r_{mn} J_m \bar{J}_n \,, \tag{22}$$

$$\bar{U}_E = \frac{1}{2\omega^2}\sum_{m=1}^{n_m}\sum_{n=1}^{n_m} s_{mn} J_m \bar{J}_n \,, \tag{23}$$

$$\bar{U}_M = \frac{1}{2}\sum_{m=1}^{n_m}\sum_{n=1}^{n_m} l_{mn} J_m \bar{J}_n \,. \tag{24}$$

These quadratic forms (=homogeneous polynomials of the 2^{nd} degree) in the complex mesh currents are real and *never negative (by their very nature); in fact, if all the meshes have some resistance, elastance, and inductance, these forms are always positive unless all the* J_m *vanish simultaneously. For this reason they are called* *__positive definite__* *__quadratic__* *__forms__* *in the* complex *mesh currents* J_m.

__Problem 5.__ *If a network in the sinusoidal state at the angular frequency* ω *is considered as a network of* n_e *elements of the general* parallel type (shown in Fig. 2, above), for which we have (cf. Ch. V, §2):

$$I_k = I_{cs_k} + I_{vs_k} + G_k V_k + i\omega C_k V_k + \sum_{l=1}^{n_e} \Gamma_{kl} V_l / i\omega \quad (k=1,\dots,n_e), \tag{25}$$

show that the total complex power supplied by the sources is:

$$\begin{aligned} P &= \sum_{k=1}^{n_e} (-\overline{V}_k)(I_{cs_k} + I_{vs_k}) = \sum_{k=1}^{n_e} G_k |V_k|^2 + i\omega \sum_{k=1}^{n_e} C_k |V_k|^2 + \frac{1}{i\omega} \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} \Gamma_{kl} \overline{V}_k V_l \\ &= \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} Y_{kl} \overline{V}_k V_l = \sum_{m=1}^{n_n'} \sum_{n=1}^{n_n'} y_{mn} \overline{U}_m U_n = \overline{W}_H + 2i\omega(\overline{U}_E - \overline{U}_M), \end{aligned} \tag{26}$$

where the Y_{kl} are the element admittances (cf. Ch. V, §2, eqs. 5), the y_{mn} are the nodal admittances (cf. Ch. VII, §2), while $\overline{W}_H$ is the total mean power dissipated into heat by the (resistors of the) network, $\overline{U}_E$ is the total mean energy stored in the electric field associated with (the condensers of) the network, and $\overline{U}_M$ is the total mean energy stored in the magnetic field associated with (the coils of) the network. Finally, in terms of the nodal conductances g_{mn}, the nodal capacitances c_{mn}, and the nodal invertances γ_{mn} (cf. Ch. VII, §2, Prob. 3), show that we have:

$$\overline{W}_H = \sum_{k=1}^{n_e} G_k |V_k|^2 = \sum_{m=1}^{n_n'} \sum_{n=1}^{n_n'} g_{mn} \overline{U}_m U_n , \tag{27}$$

$$\overline{U}_E = \frac{1}{2} \sum_{k=1}^{n_e} C_k |V_k|^2 = \frac{1}{2} \sum_{m=1}^{n_n'} \sum_{n=1}^{n_n'} c_{mn} \overline{U}_m U_n , \tag{28}$$

$$\overline{U}_M = \frac{1}{2\omega^2} \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} \Gamma_{kl} \overline{V}_k V_l = \frac{1}{2\omega^2} \sum_{m=1}^{n_n'} \sum_{n=1}^{n_n'} \gamma_{mn} \overline{U}_m U_n . \tag{29}$$

These quadratic forms in the complex potentials U_n $(n=1,2,\dots,n_n')$ are real and never negative (by their very nature); not considering negative conductances, capacitances, or self-invertances, of course, which would be equivalent to sources, in a sense. Actually, if all the self-nodal parameters are non-vanishing, these quadratic forms will always be positive unless all the nodal potential vanish simultaneously; i.e., they are then positive definite quadratic forms.

§7. THE CASE OF EXPONENTIALLY MODULATED SINUSOIDS.

If the one-to-one correspondence between exponentially modulated sinusoids and complex numbers, namely (cf. Ch. IV, §6):

$$A\, e^{\sigma t} \sin(\omega t + \alpha) \longleftrightarrow K A \underline{/\alpha} , \tag{1}$$

(for fixed σ, $\omega \neq 0$, and $K \neq 0$) is used instead of the usual correspondence between sinusoids and complex numbers, the transformed equations of a network, assuming that all the currents and voltages are now exponentially modulated sine functions (with the same σ and ω), may be obtained from the usual complex equations for the sinusoidal *state of the network simply by substituting the complex (or generalized angular) frequency $p = \sigma + i\omega$ for the usual $i\omega$, and of course, interpreting the complex currents and voltages as the complex numbers corresponding to the exponentially modulated sinusoidal currents and voltages in accordance with relation (1). Naturally, these transformed equations of the network in the damped harmonic state (speaking algebraically) can also be obtained directly from the original integro-differential equations (cf. Ch. III) of the network by replacing differentiation by multiplication by p, and integration by division by p, and all exponentially modulated sinusoidal currents and voltages by the corresponding complex numbers according to (1).*

In this way we get the following system of equations:

$$\begin{gathered} V_k = R_k I_k + S_k I_k/p + \sum_{l=1}^{n_e} pL_{kl} I_l - E_k - D_k \quad (k=1,2,\dots,n_e), \\ \sum_{k=1}^{n_e} (k,n) I_k = 0 \quad (n=1,2,\dots,n_n'), \qquad \sum_{k=1}^{n_e} [k,m] V_k = 0 \quad (m=1,2,\dots,n_m), \end{gathered} \tag{2}$$

for an arbitrary network of n_e general series elements in the exponentially modulated sinusoidal (or damped harmonic) state, interconnected into n_n nodes in n_c components, with $n_n' = n_n - n_c$ independent nodes, and $n_m = n_e - n_n'$ independent meshes. (The rest of the notation should be clear by now.)

As done above in §5 for the sinusoidal state, we also obtain here:

$$\sum_{k=1}^{n_e} \overline{V}_k I_k = 0, \qquad \text{and also:} \qquad \sum_{k=1}^{n_e} V_k \overline{I}_k = 0, \tag{3}$$

and the summations may be limited to the elements of any given component (=maximal separate part) of the network.

From the second of eqs. (3) we obtain:

$$\sum_{k=1}^{n_e} (E_k \overline{I}_k + D_k \overline{I}_k) = \sum_{k=1}^{n_e} R_k |I_k|^2 + \frac{1}{p} \sum_{k=1}^{n_e} S_k |I_k|^2 + p \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} L_{kl} I_k I_l . \tag{4}$$

The following notation, borrowed from classical mechanics, is now quite standard:

$$2F = \sum_{k=1}^{n_e} R_k |I_k|^2 , \tag{5}$$

$$V = \frac{1}{2|p|^2} \sum_{k=1}^{n_e} S_k |I_k|^2 = \frac{1}{2} \sum_{k=1}^{n_e} S_k |Q_k|^2 \qquad (I_k = pQ_k) \tag{6}$$

$$T = \frac{1}{2} \sum_{k=1}^{n_e} \sum_{l=1}^{n_e} L_{kl} \overline{I}_k I_l ; \tag{7}$$

these are called the dissipation, potential, and kinetic energy functions, respectively, just as names. They are functions of p since the I_k are. (They may also be called the resistive, capacitive, and inductive, energy functions, respectively, or thermic, electric, and magnetic energy functions.)

The first two of these quadratic functions of the I_k $(k=1,\dots,n_e)$ are clearly real and never negative, no matter what complex values the I_k may have (not considering, of course, negative resistances or elastances, which in a sense would be equivalent to sources).

The same is true of the third quadratic function (7), since it is the same function as that of the corresponding network in the sinusoidal state, for which it is true.

By substituting $I_k = \sum_{m=1}^{n_m} [k,m] J_m$, in terms of the complex mesh quantities J_m $(m=1,2,\ldots,n_m)$, the above real non-negative energy quadratic forms become:

$$2F = \sum_{m=1}^{n_m}\sum_{n=1}^{n_m} r_{mn} J_m \bar{J}_n, \qquad V = \frac{1}{2|p|^2}\sum_{m=1}^{n_m}\sum_{n=1}^{n_m} s_{mn} J_m \bar{J}_n, \qquad T = \frac{1}{2}\sum_{m=1}^{n_m}\sum_{n=1}^{n_m} l_{mn} J_m \bar{J}_n, \quad (8)$$

where (as usual) the mesh parameters r_{mn}, s_{mn}, l_{mn}, are given by (cf. Ch. VI, §2):

$$r_{mn} = \sum_{k=1}^{n_e} [k,m] R_k [k,n], \quad s_{mn} = \sum_{k=1}^{n_e} [k,m] S_k [k,n], \quad l_{mn} = \sum_{k=1}^{n_e}\sum_{l=1}^{n_e} [k,m] L_{kl} [l,n]; \quad (9)$$

and, actually, all the quadratic forms (8) will be positive for all J_m (unless all the J_m vanish simultaneously) if all the $r_{mm} s_{mm} l_{mm} \neq 0$.

The <u>complex</u> (or generalized ~~angular~~) <u>frequencies</u> <u>of</u> <u>free</u> <u>oscillation</u>, i.e., the <u>natural</u> <u>complex</u> <u>frequencies</u> of a network are the possible values that the complex frequency <u>p</u> may have in order that the network have non-vanishing response quantities when all its sources are nullified. Accordingly, by eqs. (4), such natural complex frequencies <u>p</u> must satisfy the following equation:

$$F(p) + \bar{p}\,V(p) + p\,T(p) = 0, \qquad (10)$$

where $F(p)$, $V(p)$, and $T(p)$ are the real non-negative values of the energy functions for the value of <u>p</u> in question.

Putting $p = \sigma + i\omega$ and $\bar{p} = \sigma - i\omega$ in eq. (10) we obtain:

$$F + \sigma(V+T) = 0 \qquad \& \qquad \omega(T - V) = 0. \qquad (11)$$

From these we see that <u>p</u> cannot be real and positive ($p=\sigma>0$ & $\omega=0$) unless (numerically) $F=V=T=0$ (a case of no practical importance). Furthermore, <u>p</u> cannot be a pure imaginary ($\sigma=0$ & $p=i\omega$) unless $F=0$. Finally, if $FVT \neq 0$, then $\sigma = -F/(V+T) < 0$, so that the real part of <u>p</u> will be negative in this case.

<u>Problem</u>. Establish the equations of an arbitrary network of general parallel elements assuming that all the (exciting and response) currents and voltages are exponentially modulated sinusoids with the same time constant $\tau = -1/\sigma$ and logarithmic decrement $-2\pi\sigma/\omega$.

<u>Note</u>. In any actual network, every element shall have some resistance, even if only very small. This means that <u>F</u> shall never vanish, unless all the currents vanish (not considering negative resistances, of course); i.e., <u>F</u> is a positive definite quadratic form. This implies that no natural complex frequency of any actual network can be a pure imaginary, which means that the network can have no real natural frequency. Consequently, the determinant of the complex equations of an actual network in the sinusoidal state cannot vanish; and this implies the existence and uniqueness theorems for the currents and voltages in any actual network in the sinusoidal state. When negative resistances are considered, this is no longer valid, since <u>F</u> may then vanish for non-vanishing currents. In this case (e.g., in networks with vacuum tubes) spontaneous oscillations of a certain frequency may appear in the network, even if there are no sources of this frequency in the network (as long as there is some other source of energy to sustain the oscillations, of course).